Description/definitions/preconditions:

Suppose a sphere with radius R and some primary meridian where $\lambda = 0$.

I am standing on its north pole (i.e. $\theta = 0$), from where I can observe its entire surface.

The farthest observable point is my antipodal point (i.e. the south pole, where $\theta = \pi$).

The antipodal distance obviously equals $D_{\mathrm{AP}} = \pi R$.

I see things at positions (r_i, λ_i) where r_i is measured along a geodesic at the surface.

It should be obvious that $\theta_i = \frac{r_i}{R}$.

We define dimensionless distance as $\rho \coloneqq \frac{r}{D_{\mathrm{AP}}} = \frac{r}{\pi R} = \frac{\theta}{\pi}$, hence $\theta = \pi\rho$.

Any position on the sphere is $P_i \coloneqq (\rho_i, \lambda_i)$, where $0 \le \rho_i \le 1$ and $0 \le \lambda_i < 2\pi$.

Situation:

I am observing two fixed things, positioned at $P_1 = (\rho_1, \lambda_1)$ and $P_2 = (\rho_2, \lambda_2)$.

Questions:

Q1: What is the great circle through P_1 and P_2 in terms of $\{\rho_1, \lambda_1, \rho_2, \lambda_2\}$?

Q2: What is the shortest dimensionless distance (or angle as seen from the centre) from $P_0 = (\rho_0, \lambda_0)$ to this great circle?

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Answers:

A1: We'll initially use (ϕ, λ) for (*latitude*,*longitude*), since that is the most intuitive.

A unit sphere $(R = 1)$ around $\mathcal{O}$ in cartesian coordinates is given by:

$$x = \cos\phi\cos\lambda\,,\ \ y = \cos\phi\sin\lambda\,,\ \ z = \sin\phi\,;$$

a great circle also lies in a plane through $\mathcal{O}$:

$$ax + by + cz = 0\,;$$

and it contains: $P_1 = (\phi_1, \lambda_1) = (\cos\phi_1\cos\lambda_1\,, \cos\phi_1\sin\lambda_1\,, \sin\phi_1)$

as well as: $P_2 = (\phi_2, \lambda_2) = (\cos\phi_2\cos\lambda_2\,, \cos\phi_2\sin\lambda_2\,, \sin\phi_2)$

The plane's normal vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ will then be (a multiple of) the cross product: $\overrightarrow{P_1} \times \overrightarrow{P_2}$,

which is:

$$\begin{bmatrix} y_1 z_2 - y_2 z_1 \\ z_1 x_2 - z_2 x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} \cos\phi_1\sin\lambda_1\sin\phi_2 - \cos\phi_2\sin\lambda_2\sin\phi_1 \\ \sin\phi_1\cos\phi_2\cos\lambda_2 - \sin\phi_2\cos\phi_1\cos\lambda_1 \\ \cos\phi_1\cos\lambda_1\cos\phi_2\sin\lambda_2 - \cos\phi_2\cos\lambda_2\cos\phi_1\sin\lambda_1 \end{bmatrix}$$

using:

$$\cos\beta\sin\alpha - \cos\alpha\sin\beta = \sin(\alpha - \beta)$$

we find:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} +\cos\phi_1\sin\phi_2\sin\lambda_1 - \sin\phi_1\cos\phi_2\sin\lambda_2 \\ -\cos\phi_1\sin\phi_2\cos\lambda_1 + \sin\phi_1\cos\phi_2\cos\lambda_2 \\ \cos\phi_1\cos\phi_2\sin(\lambda_2 - \lambda_1) \end{bmatrix}$$

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The equation of the plane $(\boldsymbol{a}x + \boldsymbol{b}y + \boldsymbol{c}z = 0)$ now becomes:

$$\begin{aligned}
& (+\cos\phi_1 \sin\phi_2 \sin\lambda_1 - \sin\phi_1 \cos\phi_2 \sin\lambda_2)\cos\phi\cos\lambda \\
+\ & (-\cos\phi_1 \sin\phi_2 \cos\lambda_1 + \sin\phi_1 \cos\phi_2 \cos\lambda_2)\cos\phi\sin\lambda \\
+\ & \cos\phi_1 \cos\phi_2 \sin(\lambda_2 - \lambda_1)\sin\phi \\
=\ & 0\,.
\end{aligned}$$

First 2 terms:

$$\begin{aligned}
& (+\cos\phi_1 \sin\phi_2 \sin\lambda_1 - \sin\phi_1 \cos\phi_2 \sin\lambda_2)\cos\phi\cos\lambda \\
+\ & (-\cos\phi_1 \sin\phi_2 \cos\lambda_1 + \sin\phi_1 \cos\phi_2 \cos\lambda_2)\cos\phi\sin\lambda
\end{aligned}$$

factor out $\cos\phi$:

$$\begin{aligned}
= [\ & (+\cos\phi_1 \sin\phi_2 \sin\lambda_1 - \sin\phi_1 \cos\phi_2 \sin\lambda_2)\cos\lambda \\
+\ & (-\cos\phi_1 \sin\phi_2 \cos\lambda_1 + \sin\phi_1 \cos\phi_2 \cos\lambda_2)\sin\lambda \\
]&\cos\phi
\end{aligned}$$

factor in $\cos\lambda$: and $\sin\lambda$:

$$\begin{aligned}
= [\ & +\cos\phi_1 \sin\phi_2 \sin\lambda_1 \cos\lambda - \sin\phi_1 \cos\phi_2 \sin\lambda_2 \cos\lambda \\
+\ & -\cos\phi_1 \sin\phi_2 \cos\lambda_1 \sin\lambda + \sin\phi_1 \cos\phi_2 \cos\lambda_2 \sin\lambda \\
]&\cos\phi
\end{aligned}$$

rearrange:

$$\begin{aligned}
= [\ & +\cos\phi_1 \sin\phi_2 \sin\lambda_1 \cos\lambda - \cos\phi_1 \sin\phi_2 \cos\lambda_1 \sin\lambda \\
& +\sin\phi_1 \cos\phi_2 \cos\lambda_2 \sin\lambda - \sin\phi_1 \cos\phi_2 \sin\lambda_2 \cos\lambda \\
]&\cos\phi
\end{aligned}$$

factor out:

$$\begin{aligned}
= [\ & \cos\phi_1 \sin\phi_2 (\sin\lambda_1 \cos\lambda - \cos\lambda_1 \sin\lambda) \\
+\ & \sin\phi_1 \cos\phi_2 (\cos\lambda_2 \sin\lambda - \sin\lambda_2 \cos\lambda) \\
]&\cos\phi
\end{aligned}$$

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simplify:
$$= [\quad -\cos\phi_1 \sin\phi_2 \sin(\lambda-\lambda_1) + \sin\phi_1 \cos\phi_2 \sin(\lambda-\lambda_2) \quad]\cos\phi$$

yielding:
$$[\quad -\cos\phi_1 \sin\phi_2 \sin(\lambda-\lambda_1) + \sin\phi_1 \cos\phi_2 \sin(\lambda-\lambda_2) \quad]\cos\phi + \cos\phi_1 \cos\phi_2 \sin(\lambda_2-\lambda_1)\sin\phi = 0$$
(the plane's full equation)

hence:
$$\cos\phi_1 \cos\phi_2 \sin(\lambda_2-\lambda_1)\sin\phi = [\cos\phi_1 \sin\phi_2 \sin(\lambda-\lambda_1) - \sin\phi_1 \cos\phi_2 \sin(\lambda-\lambda_2)]\cos\phi$$

Because I am standing on the north pole, it is easier to use the colatitude $\theta = \frac{\pi}{2} - \phi$, which reflects the distance along the surface via $\theta = \pi\rho$.

substituting: $\phi \to \left(\frac{\pi}{2} - \theta\right)$, hence: $\cos\phi = \sin\theta$ and: $\sin\phi = \cos\theta$

renders:
$$\sin\theta_1 \sin\theta_2 \sin(\lambda_2-\lambda_1)\cos\theta = [\sin\theta_1 \cos\theta_2 \sin(\lambda-\lambda_1) - \cos\theta_1 \sin\theta_2 \sin(\lambda-\lambda_2)]\sin\theta$$

or:
$$\frac{\cos\theta}{\sin\theta} = \frac{\sin\theta_1 \cos\theta_2 \sin(\lambda-\lambda_1) - \cos\theta_1 \sin\theta_2 \sin(\lambda-\lambda_2)}{\sin\theta_1 \sin\theta_2 \sin(\lambda_2-\lambda_1)}$$

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hence: $$\cot \pi\rho = \frac{\sin \pi\rho_1 \cos \pi\rho_2 \sin(\lambda-\lambda_1) - \cos \pi\rho_1 \sin \pi\rho_2 \sin(\lambda-\lambda_2)}{\sin \pi\rho_1 \sin \pi\rho_2 \sin(\lambda_2-\lambda_1)}$$

so: $$\rho = \frac{1}{\pi} \operatorname{arccot} \frac{\sin \pi\rho_1 \cos \pi\rho_2 \sin(\lambda-\lambda_1) - \cos \pi\rho_1 \sin \pi\rho_2 \sin(\lambda-\lambda_2)}{\sin \pi\rho_1 \sin \pi\rho_2 \sin(\lambda_2-\lambda_1)}$$

Algorithm for dimensionless distance in direction λ
from north pole to great circle through (ρ_1, λ_1) and (ρ_2, λ_2) :

function $\rho(\lambda; \rho_1, \lambda_1, \rho_2, \lambda_2)$
{ $\theta_1 = \pi\rho_1$, $u_1 = \sin\theta_1$, $v_1 = \cos\theta_1$;
$\theta_2 = \pi\rho_2$, $u_2 = \sin\theta_2$, $v_2 = \cos\theta_2$;
$x = u_1 v_2 \sin(\lambda - \lambda_1) - u_2 v_1 \sin(\lambda - \lambda_2)$;
$y = u_1 u_2 \sin(\lambda_2 - \lambda_1)$;
$\theta = \frac{\pi}{2} - \arctan(y/x)$; // "atan" in most computer languages
return θ/π ; // dimensionless distance
};

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A2: The angle β between P_0 and the great circle through P_1 and P_2 equals $\frac{\pi}{2} - \alpha$, where α is the angle between the vector $\overrightarrow{P_0}$ and this great circle's normal vector, the latter being the cross product of $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$. Then α, hence β, is found via the inner product:

$$\overrightarrow{P_0} \cdot \left(\overrightarrow{P_1} \times \overrightarrow{P_2}\right) = \left|\overrightarrow{P_0}\right| \cdot \left|\overrightarrow{P_1} \times \overrightarrow{P_2}\right| \cdot \cos\alpha = \left|\overrightarrow{P_0}\right| \cdot \left|\overrightarrow{P_1} \times \overrightarrow{P_2}\right| \cdot \sin\beta .$$

We turn them into unit vectors: $\hat{P} = \begin{bmatrix} \sin\theta\cos\lambda \\ \sin\theta\sin\lambda \\ \cos\theta \end{bmatrix}$, yielding:

$$\widehat{P_0} \cdot \left(\widehat{P_1} \times \widehat{P_2}\right) = \left|\widehat{P_1} \times \widehat{P_2}\right| \cdot \cos\alpha = \left|\widehat{P_1} \times \widehat{P_2}\right| \cdot \sin\beta$$

$$\therefore \beta = \arcsin\frac{\widehat{P_0}\cdot\left(\widehat{P_1}\times\widehat{P_2}\right)}{\left|\widehat{P_1}\times\widehat{P_2}\right|}$$

$$\widehat{P_1} \times \widehat{P_2} = \begin{bmatrix} \sin\theta_1\cos\lambda_1 \\ \sin\theta_1\sin\lambda_1 \\ \cos\theta_1 \end{bmatrix} \times \begin{bmatrix} \sin\theta_2\cos\lambda_2 \\ \sin\theta_2\sin\lambda_2 \\ \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \sin\theta_1\sin\lambda_1\cos\theta_2 - \sin\theta_2\sin\lambda_2\cos\theta_1 \\ \cos\theta_1\sin\theta_2\cos\lambda_2 - \cos\theta_2\sin\theta_1\cos\lambda_1 \\ \sin\theta_1\cos\lambda_1\sin\theta_2\sin\lambda_2 - \sin\theta_2\cos\lambda_2\sin\theta_1\sin\lambda_1 \end{bmatrix}$$

I really don't want to work this out any further in a purely mathematical way!

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Algorithmic (using arrays with zero-based indices):

function vectorLength(V) $\{L^2 = 0$; for $(i$ in $V)$ $\left\{L^2 \mathrel{+}= V_i^2\right\}$; return $\sqrt{L^2}$;};

function innerProduct(Q, R) $\{R = 0$; for $(i$ in Q & $R)$ $\{R \mathrel{+}= Q_i R_i\}$; return R;};

function crossProduct(Q, R) {return $[Q_1R_2 - Q_2R_1, Q_2R_0 - Q_0R_2, Q_0R_1 - Q_1R_0]$;};

function distanceToGreatCircle(P_0, P_1, P_2)

{ $\rho_0 = P_0["\rho"]\,,\ \theta_0 = \pi\rho_0\,,\ \lambda_0 = P_0["\lambda"]\,;$

$\rho_1 = P_1["\rho"]\,,\ \theta_1 = \pi\rho_1\,,\ \lambda_1 = P_1["\lambda"]\,;$

$\rho_2 = P_2["\rho"]\,,\ \theta_2 = \pi\rho_2\,,\ \lambda_2 = P_2["\lambda"]\,;$

$u_0 = \sin\theta_0\,, \quad x_0 = u_0\cos\lambda_0\,, \quad y_0 = u_0\sin\lambda_0\,, \quad z_0 = \cos\theta_0\,;$

$u_1 = \sin\theta_1\,, \quad x_1 = u_1\cos\lambda_1\,, \quad y_1 = u_1\sin\lambda_1\,, \quad z_1 = \cos\theta_1\,;$

$u_2 = \sin\theta_2\,, \quad x_2 = u_2\cos\lambda_2\,, \quad y_2 = u_2\sin\lambda_2\,, \quad z_2 = \cos\theta_2\,;$

$\overrightarrow{P_{12}} =$ crossProduct$([x_1, y_1, z_1], [x_2, y_2, z_2])$;

$P_{012} =$ innerProduct$\left([x_0, y_0, z_0], \overrightarrow{P_{12}}\right)$;

$L_{12} =$ vectorLength$\left(\overrightarrow{P_{12}}\right)$;

return $\frac{1}{\pi}\arcsin(P_{012}/L_{12})$; // dimensionless distance

};

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Und jetzt kommen die Kamele! *Zet's Churman and ze Churmans make no chokes, but to house have zey it also: a port**ai**ble radio...* It litterally means: *And now, the camels are coming!* and it can be interpreted as: *The clue will now be revealed!* or: *Surprise, surprise!* but also: *Houston, we have a problem!*

Instead of living on a 2-sphere (the 2-surface of a normal (3D) ball), we now pretend to be on/in a 3-sphere, a glome, which is the 3-surface of a 4-dimensional hypersphere.
Our looking direction will now consist of twe angles: $\lambda =$ *longitude* & $\varphi =$ *latitude*, which we normalise: $0 \leq \lambda < 2\pi$ and $\frac{-\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, thus making them fully equivalent to *right ascension* and *declination* in astronomy (but in proper units, e.g. radians instead of sexagesimal degrees, let alone those stupid hours, minutes & seconds for R.A. [which are historically explainable, OK]).
If $\varphi < 0$ you look down, if $\varphi > 0$ you look up, if $\varphi = 0$ you look forward (and with this domain of λ you'll turn your head left and "lefter" and "leftest" until it made a full rotation 😄).
We now observe things at distances in various directions at: $P_i \coloneqq (\rho_i, \lambda_i, \varphi_i)$, where ρ is dimensionless as before. Like a "normal" sphere, a glome has an antipodal distance of πR and we still have $\theta = \pi\rho$. Any point of a unit glome ($R_{\text{hyp}} = 1$) is fully defined by $(\theta, \lambda, \varphi)$.

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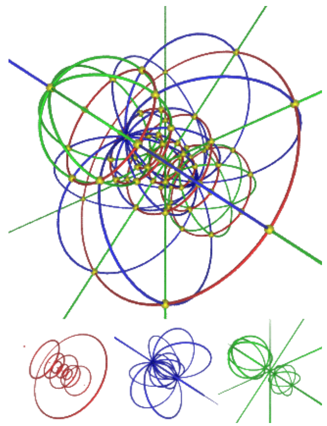
And then it is the same story as before:

Q1': What is the great circle through P_1 and P_2 in terms of $\{\rho_1, \lambda_1, \varphi_1, \rho_2, \lambda_2, \varphi_2\}$?

It is the equivalent of a straight line in Euclidean space, so there should be one and only one such great circle.

Q2': What is the shortest dimensionless distance (or angle as seen from the hypercentre) from $P_0 = (\rho_0, \lambda_0, \varphi_0)$ to this great circle?

My actual problem is as follows. SUPPOSE the cosmos is a 3-sphere (which it actually is, see http://henk-reints.nl/astro/HR-Geometry-of-universe-slideshow.pdf). YOU are at $P_1 = (\rho_1, \lambda_1, \varphi_1)$ (i.e. distance, right ascension, declination) as observed by ME. We both observe some celestial object (e.g. the distant Yonder galaxy), which is seen by me at $P_2 = (\rho_2, \lambda_2, \varphi_2)$. I also observe some object at $P_0 = (\rho_0, \lambda_0, \varphi_0)$. How far is it away from YOUR line of sight to P_2 ?

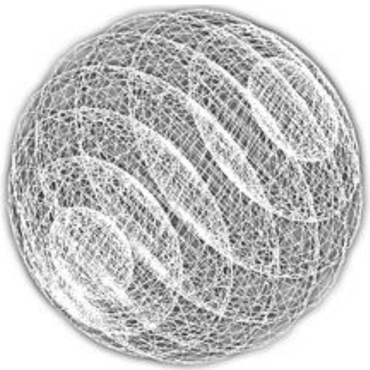


I have daltonism!

*Is it **my** fault that I cannot properly envisage a glome using a 2-dimensional projection of a 3-dimensional projection of the damned thing? As if one tries to explain a ball to a flatlander by showing him a rod.*

*I **do** however realise very well that **all** my lines of sight in **any** direction come together in my antipodal point, which I thus perceive as a large 2-sphere around me.*

There is a non-negligible lens effect, especially for distant objects beyond the "equator", which is halfway the AP.



Can YOU distinguish front and rear?

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A circle of latitude on the southern hemisphere with a radius measured from the North Pole also is a circle around the South Pole with a radius measured from there. Its *inner* area as seen from the North is its *outer* area as seen from the South and v.v. Together, they equal Earth's total surface area.

3-spherical cosmos: a large ball around us with a radius exceeding half the antipodal distance also is a smaller ball around the antipodal point. Its *inner* volume as seen from here is its *outer* volume as seen from there and v.v. Together, they equal the total volume of the cosmos. We observe the outside of this smaller ball in all directions. When making a 360° turn where you stand, you appear to orbit the thing as seen from there.

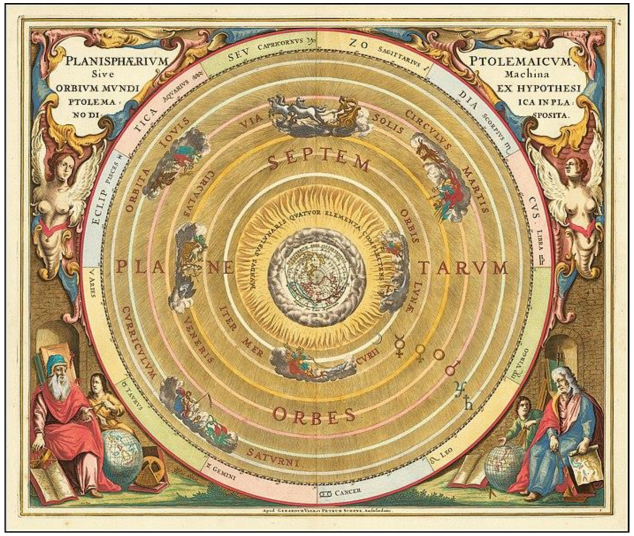

Image: Ptolemy's geocentric epicyclic model of the cosmos.

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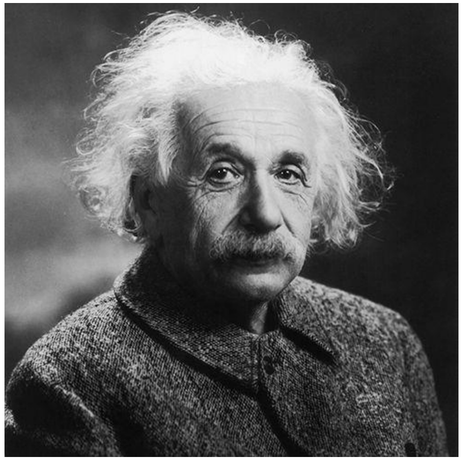
As seen from the North Pole, a point of an arbitrary great circle (i.e. a geodesic) not going through the poles is "seen" in ***ALL*** directions (i.e. longitudes).

Similarly, a 3-spherical geodesic has a sky projection that may look weird and differ significantly from Euclidean. Objects at strange locations in the sky can actually be near it.

Naive application of Euclidean geometry as if the geodesic were a straight line in a flat universe is completely beside reality.



"Einstein's razor":

Everything should be made as simple as possible, but not simpler.

(But he probably never said this).

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A1': A unit glome can be represented by (see https://en.wikipedia.org/wiki/3-sphere):

$x \quad = \sin\psi' \sin\theta' \cos\varphi'$ *longitude*: $\varphi' \in [0,2\pi)$

$y \quad = \sin\psi' \sin\theta' \sin\varphi'$

$z \quad = \sin\psi' \cos\theta'$ *co**la**titude*: $\theta' \in [0,\pi]$

$w \quad = \cos\psi'$ *co**al**titude*: $\psi' \in [0,\pi]$ *distance*: $\rho = \psi'/\pi$

I prefer to use

$\theta = \pi\rho$ instead of ψ' (*coaltitude* ∝ *distance*)

& $\varphi = \frac{\pi}{2} - \theta'$ (*declination*) $\therefore \cos\theta' = \sin\varphi$ & $\sin\theta' = \cos\varphi$

& λ instead of φ' (*right ascension*)

yielding:

$x \quad = \sin\pi\rho \cos\varphi \cos\lambda$

$y \quad = \sin\pi\rho \cos\varphi \sin\lambda$

$z \quad = \sin\pi\rho \sin\varphi$

$w \quad = \cos\pi\rho$

We have two known points:

$$P_1 = (\rho_1, \lambda_1, \varphi_1) \therefore \widehat{P_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix}$$ (using the above)

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and: $P_2 = (\rho_2, \lambda_2, \varphi_2) \therefore \widehat{P_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix}$

We want to find the distance to this geodesic through P_1 & P_2 as seen from the "north pole" (MY location),

i.e.: $\rho(\lambda, \varphi; \rho_1, \lambda_1, \varphi_1, \rho_2, \lambda_2, \varphi_2)$

Together, P_1 & P_2 define a 2-plane that also contains the origin. The geodesic must be in it, as well as on the glome, of course. This plane is given by:

$$\vec{P}(a, b) = a\widehat{P_1} + b\widehat{P_2} \quad \text{where } a, b \in \mathbb{R}$$

The geodesic must obviously be a unit circle around the origin.

Hmm, I could have used this same approach for the geodesic on a 2-sphere...

We want to define this circle using two orthonormal vectors,

so we must now find $\widehat{P_3} = a\widehat{P_1} + b\widehat{P_2}$ such that $\widehat{P_3} \perp \widehat{P_1}$

the projection of $\widehat{P_2}$ on $\widehat{P_1}$ equals $\widehat{P_1} \cos\alpha$

and the inner product yields $\cos\alpha = \widehat{P_1} \cdot \widehat{P_2}$

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Subtraction of this projection: $\left(\widehat{P_1} \cdot \widehat{P_2}\right)\widehat{P_1}$

from: $\widehat{P_2}$

yields: $\vec{Q} = \widehat{P_2} - \left(\widehat{P_1} \cdot \widehat{P_2}\right)\widehat{P_1}$,

which we normalise to a unit vector: $\widehat{P_3} = \vec{Q}/\left|\vec{Q}\right|$

We could completely work this out, but the above suffices for writing an algorithm that runs on a computer, which we will ultimately do when applying all this to the real thing.

We now have the circle: $\hat{P}(u, v) = u\widehat{P_1} + v\widehat{P_3}$ where: $u^2 + v^2 = 1$

in which we can substitute: $u = \cos\eta$, $v = \sin\eta$,

where η is the angle within the circle, starting at P_1 and going towards P_3 .

Of course it yields: $\hat{P}(\eta) = \cos\eta\,\widehat{P_1} + \sin\eta\,\widehat{P_3}$

which expands to:

$$\hat{P} = \begin{bmatrix} \sin\pi\rho\cos\varphi\cos\lambda \\ \sin\pi\rho\cos\varphi\sin\lambda \\ \sin\pi\rho\sin\varphi \\ \cos\pi\rho \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix}\cos\eta + \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{bmatrix}\sin\eta$$

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from which we find (with atan2 as available in many computer languages):

$$\pi\rho = \arccos w$$
$$\lambda = \operatorname{atan2}(y, x) \qquad (\mathrm{mod}\ 2\pi) \qquad \text{if } (\lambda < 0) \text{ then } \lambda += 2\pi;$$
$$\varphi = \arcsin(z / \sin \pi\rho)$$

A2': The angle β between $\widehat{P_0}$ and $\hat{P}$

is found via the inner product: $\hat{P} \cdot \widehat{P_0} = |\hat{P}| \cdot |\widehat{P_0}| \cdot \cos\beta$

since both are unit vectors we find: $\beta = \arccos(\hat{P} \cdot \widehat{P_0})$

We have:

$$\begin{aligned}
\hat{P} \cdot \widehat{P_0} &= x_0(x_1 \cos\eta + x_3 \sin\eta) \\
&+ y_0(y_1 \cos\eta + y_3 \sin\eta) \\
&+ z_0(z_1 \cos\eta + z_3 \sin\eta) \\
&+ w_0(w_1 \cos\eta + w_3 \sin\eta) \\
&= (x_0x_1 + y_0y_1 + z_0z_1 + w_0w_1) \cos\eta \\
&+ (x_0x_3 + y_0y_3 + z_0z_3 + w_0w_3) \sin\eta \\
&= A\cos\eta + B\sin\eta \qquad \text{where: } A = \hat{P} \cdot \widehat{P_1} \ \& \ B = \hat{P} \cdot \widehat{P_3}
\end{aligned}$$

yielding: $\beta(\eta) = \arccos(A\cos\eta + B\sin\eta)$

and: $$\frac{d\beta(\eta)}{d\eta} = \frac{A\sin\eta - B\cos\eta}{\sqrt{1-(A\cos\eta + B\sin\eta)^2}}$$

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This derivative equals zero if: $A \sin\eta = B \cos\eta$

hence: $\eta = \arctan(B/A) + n\pi \pmod{2\pi}$

which has 2 solutions satisfying: $\eta \in [0,2\pi)$

The extreme distances then are: $\beta_0 = \beta(\eta_0)\,, \quad \rho_0 = \beta_0/\pi$

as well as: $\beta_1 = \beta(\eta_1)\,, \quad \rho_1 = \beta_1/\pi$

of which we need the smallest. *This is comparable to the north and southward distances from a point on Earth to the equator; the longer one is over one of the poles*.

Now we want to find the angle between two geodesics on a 3-sphere at their point(s) of intersection. It equals the angle between the two planes in which these geodesics reside. Therefore P_{2a} and P_{2b} (which together with $\widehat{P_1}$ yield the orthonormal $\widehat{P_{3a}}$ and $\widehat{P_{3b}}$) are observed from P_1 at an angle of: $\alpha = \arccos\left(\widehat{P_{3a}} \cdot \widehat{P_{3b}}\right)$.

The absolute size of an object at distance ρ is derived from its observed angular size α via the geodetic arc between: $P_1(\rho, 0,0)$ and $P_2(\rho, \alpha, 0)$ (as if α is along the equator). We simply find $\beta = \arccos\left(\widehat{P_1} \cdot \widehat{P_2}\right)$ yielding $\frac{\beta}{\pi}$ as dim.less abs. size and $\frac{\beta}{\pi} D_{\mathrm{AP}}$ as true size.

For completeness, next page contains some more stuff about glomes (i.e. 3-spheres).

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Note: according to Einstein's general relativity, light travels along geodesics, implying the very far ends of *all* lines of sight in *each* and *every* direction coincide at the observer's antipodal point. They are converging 1-meridians, rendering a significant lensing effect at great distances.

3-volume of a ball around us:	$\frac{V}{D_{\mathrm{AP}}^3} = \frac{2\pi\rho - \sin 2\pi\rho}{\pi^2}$	cosmos:	$V_{\mathrm{U}} = \frac{2D_{\mathrm{H}}^3}{\pi}$
2-surface of the same:	$\frac{A}{D_{\mathrm{AP}}^2} = \frac{4\sin^2 \pi\rho}{\pi}$		
1-circumference thereof:	$\frac{C}{D_{\mathrm{AP}}} = 2\sin \pi\rho$		
2-circumference of glome = 2-surface of 2-equator:	$\frac{A}{D_{\mathrm{AP}}^2} = \frac{4}{\pi}$	cosmos:	$A_{\mathrm{U}} = \frac{4D_{\mathrm{H}}^2}{\pi}$
1-circumference of glome = 1-equator:	$\frac{C}{D_{\mathrm{AP}}} = 2$	cosmos:	$C_{\mathrm{U}} = 2D_{\mathrm{H}}$

Beyond the 2-equator, a ball around us is actually a smaller ball with complementary radius $(D_{AP} - r)$ around the antipodal point, of which we observe its "outer volume".

Cf. Earth's surface from the north pole to a circle of latitude on the southern hemisphere. The latter is either a large circle around the north pole with r measured from N beyond the equator (and a way too small circumference), or a small circle around the south pole with r measured from there.

Please think about next: which part of the earth is inside or outside a circle of latitude?

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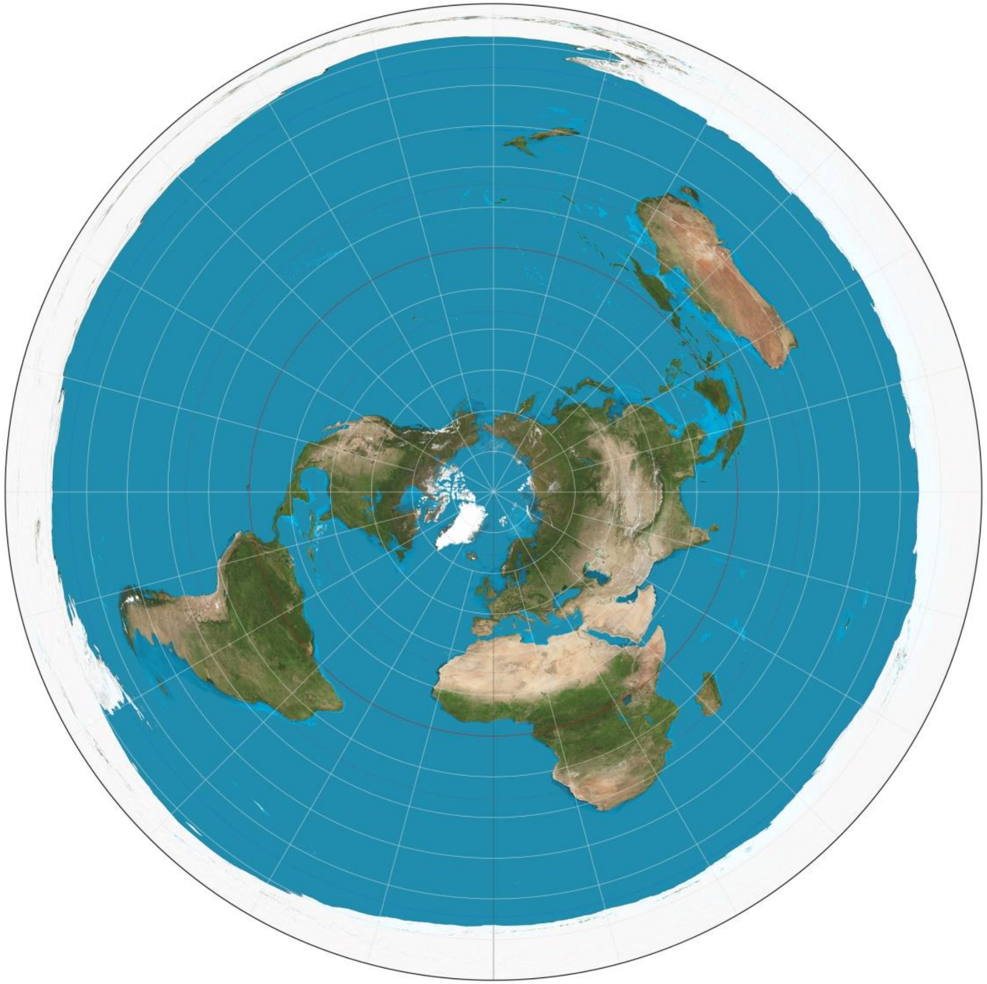
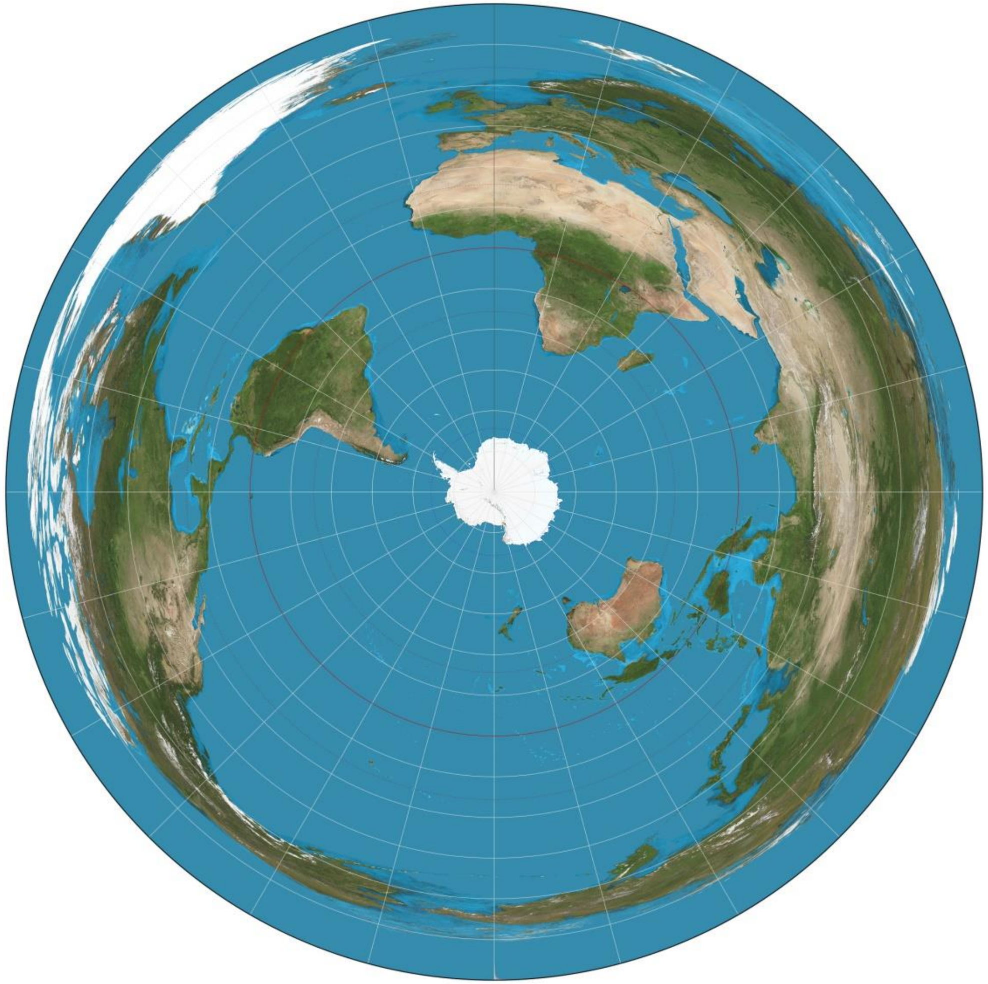
Azimuthal equidistant projections of the Earth:





Seen from the North Pole , the South Pole is a circle around the whole world.
Seen from the South Pole , the North Pole is a circle around the whole world.
Actually, both poles are single points.

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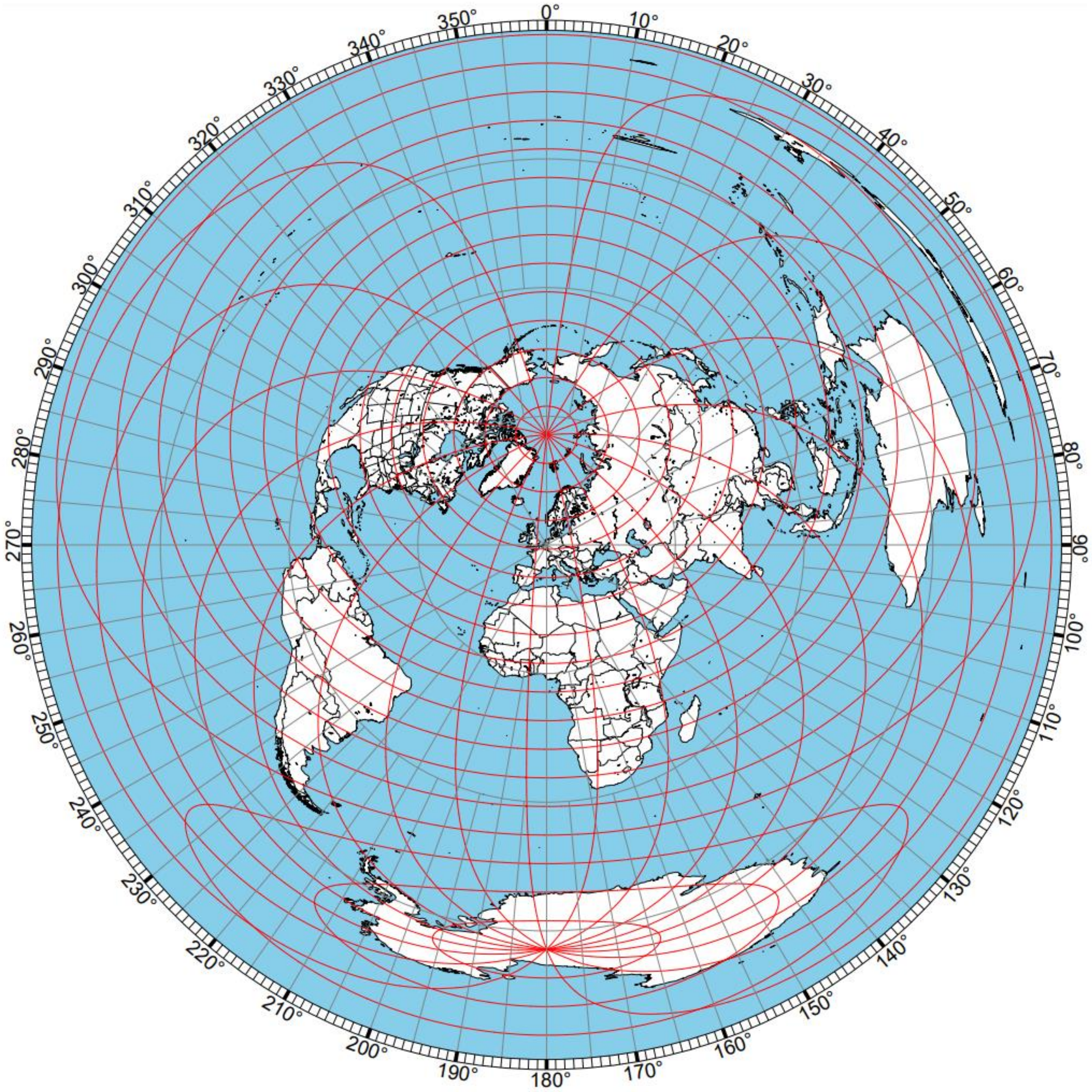
Azimuthal equidistant world map around

Helmond (NL)

51°28'42"N, 5°39'41"E,

the town where I grew up.



Map by
https://ns6t.net/azimuth/azimuth.html

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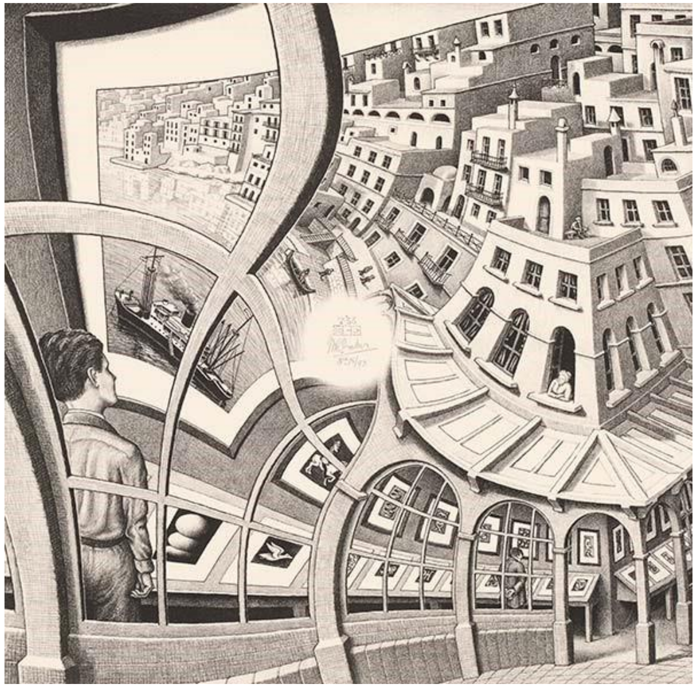

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