r :	radius of a ball or sphere;
ball:	collection of all points having a distance d to some reference point we call its centre, where $d \leq r$; it is **massive**, **inside is part of it**;
sphere:	collection of all points having a distance d to some reference point we call its centre, where $d = r$; it is **hollow**, **inside is *<u>not</u>* part of it**;
volume:	size of what is enclosed by a sphere;
surface area:	size of what encloses a ball;
	a sphere is the outer surface of a ball, a ball is a sphere plus all it encloses;
circle:	equivalent of a sphere in 2D space;
disk:	equivalent of a ball in 2D space, i.e. a circle plus all it encloses.
	A ***ball*** is ***massive***, a ***sphere*** is ***hollow***.

Single Dutch:

In *dit* document is
een "3-ball" een massieve ***ba**l* en
een "2-sphere" een holle ***bo**l*, Gijs!

Ezelsbruggetje (donkey bridgie?):
de ***a*** van ***ba**l* komt van *m**a**ssief* of ***a**lles*;
de ***o*** van ***bo**l* staat voor ***o**ppervlak*;
ook rijmt *bol* op *hol* en da's nie *vol*. *Lol*.

In daily life, a ***surface area*** is **2D** and a ***volume*** is **3D** .
In *multidimensional mathematics*, that's ~~another cook~~ different:

		cf. a desktop (no, not a computer screen, moron!)	cf. the space all around us	beyond our imagination
enclosed:	math. name:	**2-ball**	**3-ball**	**4-ball**
	conv. name:	***disk***	***ball***	hyperball
	math. size:	2-volume	3-volume	4-volume
	conv. size:	***surface area*** [m^2]	***volume*** [m^3]	hyper-volume [m^4]
		$V_2 = \pi r^2$	$V_3 = \frac{4\pi}{3} r^3$	$V_4 = \frac{\pi^2}{2} r^4$
enclosing:	math. name:	**1-sphere**	**2-sphere**	**3-sphere**
	conv. name:	***circle***	***sphere***	hypersphere
	math. size:	1-surface area	2-surface area	3-surface area
	conv. size:	***circum-ference*** [m]	***surface area*** [m^2]	volume [m^3]
	$A_n = \frac{dV_{n+1}}{dr}$	$A_1 = 2\pi r$	$A_2 = 4\pi r^2$	$A_3 = 2\pi^2 r^3$

conv. descr.	$V_n = \frac{rA_{n-1}}{n} = \frac{2\pi r^2 V_{n-2}}{n}$	$A_n = 2\pi r V_{n-1} = \frac{dV_{n+1}}{dr}$	conv. descr.
	$V_0 := 1$	$A_0 := 2$	
	$V_1 = \frac{rA_0}{1} = 2r$	$A_1 = 2\pi r V_0 = 2\pi r$	*circumf. of circle*
area of circle	$V_2 = \frac{rA_1}{2} = \pi r^2$	$A_2 = 2\pi r V_1 = 4\pi r^2$	*area of sphere*
volume of sphere	$V_3 = \frac{rA_2}{3} = \frac{4\pi}{3} r^3$	$A_3 = 2\pi r V_2 = 2\pi^2 r^3$	
	$V_4 = \frac{rA_3}{4} = \frac{\pi^2}{2} r^4$	$A_4 = 2\pi r V_3 = \frac{8\pi^2}{3} r^4$	
	$V_5 = \frac{rA_4}{5} = \frac{8\pi^2}{15} r^5$	$A_5 = 2\pi r V_4 = \pi^3 r^5$	
	$V_6 = \frac{rA_5}{6} = \frac{\pi^3}{6} r^6$	etc.	

V_n is enclosed by S_{n-1}, S_n encloses V_{n+1}
(S_n denotes the (hollow) n-sphere as such, A_n is the value of its surface area).

On the internet they often use S_n or A_n to indicate the surface of an n-ball
where they should actually have used S_{n-1} or A_{n-1}, which is rather cofunsign.

The **earth** is a **3-ball** & its **surface** is a **2-sphere** with ***2*** *dim., i.e. lat. & lon.*

n-volume of n-ball[6,7] and n-surface area of n-sphere[6,7] for $n \le 20$

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n \qquad A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}\text{, i.e. } A_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} r^n$$

where $\Gamma(x)$ is the so called **gamma function**[3]: $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

$$\Gamma(x+1) = x\Gamma(x) \qquad \forall (n \in \mathbb{N},\ n > 0):\ \Gamma(n) = (n-1)!$$

Volume of n-ball

V0	=	1					=	1.00000		
V1	=	2			*	r		2.00000	*	r
V2	**=**			**pi**	*****	**r^2**	**=**	**3.14159**	*****	**r^2**
V3	**=**	**(4/3)**	*****	**pi**	*****	**r^3**	**=**	**4.18879**	*****	**r^3**
V4	**=**	**(1/2)**	*****	**pi^2**	*****	**r^4**	**=**	**4.93480**	*****	**r^4**
V5	=	(8/15)	*	pi^2	*	r^5	=	5.26379	*	r^5
V6	=	(1/6)	*	pi^3	*	r^6	=	5.16771	*	r^6
V7	=	(16/105)	*	pi^3	*	r^7	=	4.72477	*	r^7
V8	=	(1/24)	*	pi^4	*	r^8	=	4.05871	*	r^8
V9	=	(32/945)	*	pi^4	*	r^9	=	3.29851	*	r^9
V10	=	(1/120)	*	pi^5	*	r^10	=	2.55016	*	r^10
V11	=	(64/10395)	*	pi^5	*	r^11	=	1.88410	*	r^11
V12	=	(1/720)	*	pi^6	*	r^12	=	1.33526	*	r^12
V13	=	(128/135135)	*	pi^6	*	r^13	=	0.910629	*	r^13
V14	=	(1/5040)	*	pi^7	*	r^14	=	0.599265	*	r^14
V15	=	(256/2027025)	*	pi^7	*	r^15	=	0.381443	*	r^15
V16	=	(1/40320)	*	pi^8	*	r^16	=	0.235331	*	r^16
V17	=	(512/34459425)	*	pi^8	*	r^17	=	0.140981	*	r^17
V18	=	(1/362880)	*	pi^9	*	r^18	=	8.21459e-2	*	r^18
V19	=	(1024/654729075)	*	pi^9	*	r^19	=	4.66216e-2	*	r^19
V20	=	(1/3628800)	*	pi^10	*	r^20	=	2.58069e-2	*	r^20

Surface area of n-sphere

A0	=	2					=	2.00000		
A1	**=**	**2**	*****	**pi**	*****	**r**	**=**	**6.28319**	*****	**r**
A2	**=**	**4**	*****	**pi**	*****	**r^2**	**=**	**12.5664**	*****	**r^2**
A3	**=**	**2**	*****	**pi^2**	*****	**r^3**	**=**	**19.7392**	*****	**r^3**
A4	=	(8/3)	*	pi^2	*	r^4	=	26.3189	*	r^4
A5	=			pi^3	*	r^5	=	31.0063	*	r^5
A6	=	(16/15)	*	pi^3	*	r^6	=	33.0734	*	r^6
A7	=	(1/3)	*	pi^4	*	r^7	=	32.4697	*	r^7
A8	=	(32/105)	*	pi^4	*	r^8	=	29.6866	*	r^8
A9	=	(1/12)	*	pi^5	*	r^9	=	25.5016	*	r^9
A10	=	(64/945)	*	pi^5	*	r^10	=	20.7251	*	r^10
A11	=	(1/60)	*	pi^6	*	r^11	=	16.0232	*	r^11
A12	=	(128/10395)	*	pi^6	*	r^12	=	11.8382	*	r^12
A13	=	(1/360)	*	pi^7	*	r^13	=	8.38970	*	r^13
A14	=	(256/135135)	*	pi^7	*	r^14	=	5.72165	*	r^14
A15	=	(1/2520)	*	pi^8	*	r^15	=	3.76529	*	r^15
A16	=	(512/2027025)	*	pi^8	*	r^16	=	2.39668	*	r^16
A17	=	(1/20160)	*	pi^9	*	r^17	=	1.47863	*	r^17
A18	=	(1024/34459425)	*	pi^9	*	r^18	=	0.885810	*	r^18
A19	=	(1/181440)	*	pi^10	*	r^19	=	0.516138	*	r^19
A20	=	(2048/654729075)	*	pi^10	*	r^20	=	0.292932	*	r^20

Equivalent (simpler?) formulas:

$$V_{2k} = \frac{\pi^k}{k!} r^{2k}$$

$$V_{2k+1} = \frac{2(2\pi)^k}{(2k+1)!!} r^{2k+1}$$

$$A_{2k-1} = \frac{dV_{2k}}{dr} = \frac{2\pi^k}{(k-1)!} r^{2k-1}$$

$$A_{2k} = \frac{dV_{2k+1}}{dr} = \frac{2(2\pi)^k}{(2k-1)!!} r^{2k}$$

where "!!" is the *double factorial*[9]:

$$7!! = 7 \times 5 \times 3 \times 1$$

$$8!! = 8 \times 6 \times 4 \times 2$$

$$n!! = n \times (n-2) \times (n-4) \times \cdots \times \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

n-sphere caps.

An n-sphere has great $(n-1)$-spheres, cf. great circles like equator & meridians on Earth. *Colatitude* ≔ angle to North Pole, *latitude* ≔ angle to equator, both along a meridian.

n-sphere cap = portion of n-sphere within a given colatitude

= non-Euclidean (concave/convex) n-ball with an $(n-1)$-surface area.

Cf. a boiled egg's top that has been cut off. Conventionally, the edible part of the egg inside this top is part of the cap. In this very document however, I mean only the cut-off part of the egg's shell to be a cap, without its content.

Cf. *surface* of the ***3**-ball* named Earth = a ***2**-sphere*, the (perfectly round) arctic *ice cap* is a ***2**-sphere cap*, which is a (non-Euclidean) ***2**-ball* (i.e. *disk*) within a given ***1**-sphere* (*circle* of latitude), the latter often identified by its *colatitude* or its *radius* measured from the North Pole as an *arc length* along a meridian on Earth's surface.

In daily life, this ice cap's ***2**-volume* & ***1**-surface area* are called *surface area* & *circumference*, respectively.

A **2**-sphere cap is a (non-Euclidean) **2**-ball (disk) with a **1**-surface (circumf.) & a **3**-sphere cap is a (non-Euclidean) **3**-ball with a **2**-surface.

In this document, I define:

V_n^{cap} := n-volume of an n-sphere cap;
A_n^{cap} := $(n-1)$-surface area of an n-sphere cap.

hence: V_2^{cap} is the **2**-vol. of a **2**-ball (disk) with a **1**-surf. area (circumference);
V_3^{cap} is the **3**-vol. of a (massive) **3**-ball with a (hollow) **2**-surf. area.

In his *introduction*, **S.Li** defines[1]:

Let S^n be an n-hypersphere, or n-sphere for short,
of radius r in n-dimensional Euclidean space.

This is however inconsistent with the standard definition of an n-sphere.

An n-sphere exists in $(n+1)$-dimensional Euclidean space.

S.Li says a **2**-sphere would exist in **2**D_{Eucl}, but since a **2**-sphere is the surface of a **3**-ball, it requires **3**D_{Eucl}. In **2**D_{Eucl} exist **1**-spheres (circles).

Earth's surface is a 2-sphere whilst Earth itself is a 3-ball in 3D_{Eucl}.
King Arthur's table top is a 2-ball in 2D_{Eucl} & its edge is a 1-sphere.

S.Li derives:

$$\boldsymbol{V_n^{\mathrm{cap}}} = \tfrac{1}{2} V_n \cdot I_{\sin^2\varphi}\left(\tfrac{n+1}{2}, \tfrac{1}{2}\right) = \quad \boldsymbol{\frac{1}{2} \cdot \frac{\pi^{n/2}}{\Gamma(1+n/2)} r^n \cdot I_{\sin^2\varphi}\left(\frac{n+1}{2}, \frac{1}{2}\right)}$$

$$\boldsymbol{A_n^{\mathrm{cap}}} = \tfrac{1}{2} A_n \cdot I_{\sin^2\varphi}\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right) = \quad \boldsymbol{\frac{\pi^{n/2}}{\Gamma(n/2)} r^{n-1} \cdot I_{\sin^2\varphi}\left(\frac{n-1}{2}, \frac{1}{2}\right)}$$

But what he would call a **3**-sphere actually is a **2**-sphere, etc.

I rename V_n^{cap} to W_n^{cap}, which then is the n-volume of an n-ball top, being a fraction of the n-volume of the entire n-ball, so it includes part of the n-ball's original interior (the edible part of the aforementioned egg). In this very document, I will further ignore W_n^{cap}. By renaming, the symbol V_n^{cap} has become available for reuse and I rename A_n^{cap} to V_{n-1}^{cap}, which would for example be the surface of an ice cap on Earth. It is the $(n-1)$-volume of an $(n-1)$-sphere cap, thus correcting for S.Li's error in the dimension.

Via an $(n-1) \to n$ transformation, V_n^{cap} now is the n-volume of an n-sphere cap, which itself is a non-Euclidean n-ball with an $(n-1)$-surface.

S.Li's: $$\boldsymbol{A_n^{\text{cap}}} = \frac{\pi^{n/2}}{\Gamma(n/2)} r^{n-1} \cdot I_{\sin^2\varphi}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

has now become: $$V_{n-1}^{\text{cap}} = \frac{\pi^{n/2}}{\Gamma(n/2)} r^{n-1} \cdot I_{\sin^2\varphi}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

hence: $$V_n^{\text{cap}}(\varphi) = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} r^{n} \cdot I_{\sin^2\varphi}\left(\frac{n}{2}, \frac{1}{2}\right)$$

is the n-volume of an n-sphere cap within colatitude φ, for example the surface of an ice cap or so.

We redefine $A_n^{\text{cap}} := \dfrac{dV_n^{\text{cap}}}{dr_n}$ as the $(n-1)$-surface area of an n-sphere cap (e.g. the circumference of an ice cap),

where $r_n =$ radius of n-ball ($= n$-sphere cap) as measured along the n-surface of the n-spere on which the cap resides, i.e. the *colatitudinal arc length* (distance from North Pole along terrestrial meridian).

What the heck is $I_{\sin^2\varphi}(a,b)$?

The *Regularised Incomplete Beta function* & φ is the *colatitude*.

And what is the *Regularised Incomplete Beta function*?

It is: $I_x(a,b) = \frac{\mathrm{B}(x;a,b)}{\mathrm{B}(a,b)}$

where $\mathrm{B}(a,b)$ is the *Beta function,* which in this definition of $I_x(a,b)$ regularises the *Incomplete Beta function* $\mathrm{B}(x;a,b)$.

$$\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and $\mathrm{B}(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$

hence $\mathrm{B}(1;a,b) = \mathrm{B}(a,b)$

After substituting $\sin^2\varphi$ for x, we obtain the behemoth $I_{\sin^2\varphi}(a,b)$ which can spawn various equations using φ .

A script performing this spawning task for some (a, b) domain yielded (using recursion rules[5] & the $I_x\left(\frac{1}{2},\frac{1}{2}\right) = \frac{2}{\pi}\arctan\frac{\sqrt{x}}{\sqrt{1-x}}$ value[2]):

$$I_{\sin^2\varphi}\left(\frac{1}{2},\frac{1}{2}\right) = \frac{2\varphi}{\pi}$$

$$I_{\sin^2\varphi}\left(\frac{2}{2},\frac{1}{2}\right) = 1 - C$$

$$I_{\sin^2\varphi}\left(\frac{3}{2},\frac{1}{2}\right) = \frac{2\varphi - \sin 2\varphi}{\pi}$$

$$I_{\sin^2\varphi}\left(\frac{4}{2},\frac{1}{2}\right) = 1 - C\left(1 + \frac{S^2}{2}\right)$$

$$I_{\sin^2\varphi}\left(\frac{5}{2},\frac{1}{2}\right) = \frac{2\varphi - \sin 2\varphi - C\left(4S^3/3\right)}{\pi}$$

$$I_{\sin^2\varphi}\left(\frac{6}{2},\frac{1}{2}\right) = 1 - C\left(1 + \frac{S^2}{2} + \frac{3S^4}{8}\right)$$

$$I_{\sin^2\varphi}\left(\frac{7}{2},\frac{1}{2}\right) = \frac{2\varphi - \sin 2\varphi - C\left(4S^3/3 + 16S^5/15\right)}{\pi}$$

$$I_{\sin^2\varphi}\left(\frac{8}{2},\frac{1}{2}\right) = 1 - C\left(1 + \frac{S^2}{2} + \frac{3S^4}{8} + \frac{5S^6}{16}\right)$$

$$I_{\sin^2\varphi}\left(\frac{9}{2},\frac{1}{2}\right) = \frac{2\varphi - \sin 2\varphi - C\left(4S^3/3 + 16S^5/15 + 32S^7/35\right)}{\pi}$$

$$I_{\sin^2\varphi}\left(\frac{10}{2},\frac{1}{2}\right) = 1 - C\left(1 + \frac{S^2}{2} + \frac{3S^4}{8} + \frac{5S^6}{16} + \frac{35S^8}{128}\right)$$

where: $S \coloneqq \sin\varphi\,,\quad C \coloneqq \cos\varphi\,,\quad 0 \le \varphi \le \frac{\pi}{2}.$

Already mentioned: *Gamma function*; some values:

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$	$\approx$	1.772 453 850 905 5158;	$\Gamma(1) = 0! =$	1;
$\Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}/2$	$\approx$	0.886 226 925 452 7579;	$\Gamma(2) = 1! =$	1;
$\Gamma\left(\frac{5}{2}\right) = 3\sqrt{\pi}/4$	$\approx$	1.329 340 388 179 137;	$\Gamma(3) = 2! =$	2;
$\Gamma\left(\frac{7}{2}\right) = 15\sqrt{\pi}/8$	$\approx$	3.323 350 970 447 842;	$\Gamma(4) = 3! =$	6;
$\Gamma\left(\frac{9}{2}\right) = 105\sqrt{\pi}/16$	$\approx$	11.631 728 396 567 448;	$\Gamma(5) = 4! =$	24;
$\Gamma\left(\frac{11}{2}\right) = 945\sqrt{\pi}/32$	$\approx$	52.342 777 784 553 52;	$\Gamma(6) = 5! =$	120;
$\Gamma\left(\frac{13}{2}\right) = 10\,395\sqrt{\pi}/64$	$\approx$	287.885 277 815 044 33;	$\Gamma(7) = 6! =$	720;
$\Gamma\left(\frac{15}{2}\right) = 135\,135\sqrt{\pi}/128$	$\approx$	1 871.254 305 797 7882;	$\Gamma(8) = 7! =$	5 040;
$\Gamma\left(\frac{17}{2}\right) = 2\,027\,025\sqrt{\pi}/256$	$\approx$	14 034.407 293 483 411;	$\Gamma(9) = 8! =$	40 320;
$\Gamma\left(\frac{19}{2}\right) = 34\,459\,425\sqrt{\pi}/512$	$\approx$	119 292.461 994 609;	$\Gamma(10) = 9! =$	362 880;

$$\boldsymbol{\Gamma}\left(\frac{\mathbf{2}\boldsymbol{k}+\mathbf{1}}{\mathbf{2}} = k + \frac{1}{2}\right) = \frac{(\mathbf{2}\boldsymbol{k}-\mathbf{1})!!}{\mathbf{2}^{\boldsymbol{k}}}\sqrt{\boldsymbol{\pi}} \qquad \boldsymbol{\Gamma}(\boldsymbol{n}) = (\boldsymbol{n}-\mathbf{1})!$$

Shown above: $$\boldsymbol{V_n^{\text{cap}}(\varphi) = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} r^n \cdot I_{\sin^2\varphi}\left(\frac{n}{2},\frac{1}{2}\right)}$$

yielding: $$\boldsymbol{V_1^{\text{cap}}(\varphi)} = \frac{\pi^{(1+1)/2}}{\Gamma\left(\frac{1+1}{2}\right)} r \cdot I_{\sin^2\varphi}\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\pi}{\Gamma(1)=(0!)=1} r \cdot \frac{2\varphi}{\pi}$$

$$= \boldsymbol{2\varphi r} =$$

arc length of circle segment where φ is from circle's NP; segment extends to both sides;

$$\boldsymbol{V_2^{\text{cap}}(\varphi)} = \frac{\pi^{(2+1)/2}}{\Gamma\left(\frac{2+1}{2}\right)} r^2 \cdot I_{\sin^2\varphi}\left(\frac{2}{2},\frac{1}{2}\right) = \frac{\pi\sqrt{\pi}}{\Gamma\left(\frac{3}{2}\right)} r^2 \cdot (1-C)$$

$$= \frac{\pi\sqrt{\pi}}{\sqrt{\pi}/2} r^2 \cdot (1-C) = \boldsymbol{2\pi r^2(1-\cos\varphi)}$$

$V_2^{\text{cap}}(\varphi = 0) = 0$ seems rather obvious;

$V_2^{\text{cap}}\left(\varphi = \frac{\pi}{2}\right) = 2\pi r^2 = \frac{4\pi r^2}{2}$ area of a hemisphere;

$$\boldsymbol{V_3^{\text{cap}}(\varphi)} = \frac{\pi^{(3+1)/2}}{\Gamma\left(\frac{3+1}{2}\right)} r^3 \cdot I_{\sin^2\varphi}\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\pi^2}{\Gamma(2)=(1!)=1} r^3 \cdot \frac{2\varphi - \sin 2\varphi}{\pi}$$

$$= \boldsymbol{\pi r^3(2\varphi - \sin 2\varphi)}$$

With: $\boldsymbol{\rho := \frac{\Delta\varphi}{\varphi_{\text{antipodal}} = \pi}}$ dimensionless distance;

we find: $\rho_{\text{cap}} = \frac{\varphi_{\text{cap}}}{\pi} = \frac{\varphi}{\pi}$ which we'll simply call ρ from now on;

hence: $\varphi = \pi\rho$

Also: $\boldsymbol{D_{\text{a}}} := D_{\text{antipodal}} = \boldsymbol{\pi r}$ ($r =$ (hyper) radius of n-sphere);

ergo: $r = \frac{D_{\text{a}}}{\pi} \therefore r^m = \frac{D_{\text{a}}^m}{\pi^m}$

so: $V_3^{\text{cap}}(\varphi) = \pi r^3(2\varphi - \sin 2\varphi) = \pi \frac{D_{\text{a}}^3}{\pi^3}(2\pi\rho - \sin 2\pi\rho)$

yielding: $\boldsymbol{\frac{V_3^{\text{cap}}(\rho)}{D_{\text{a}}^3} = \frac{2\pi\rho - \sin 2\pi\rho}{\pi^2}}$ which is dimensionless;

and: $V_3^{\text{cap}}(\rho = 1) = \frac{\boldsymbol{2D_{\text{a}}^3}}{\boldsymbol{\pi}} = 2\pi^2 r^3 = A_3(r)$ 3-area of entire 3-sphere;

Also: $V_2^{\text{cap}}(\varphi) = 2\pi r^2(1 - \cos\varphi) = 2\pi \frac{D_{\text{a}}^2}{\pi^2}(1 - \cos\pi\rho)$

hence: $\boldsymbol{\frac{V_2^{\text{cap}}(\rho)}{D_{\text{a}}^2} = \frac{2(1 - \cos\pi\rho)}{\pi}}$ also dimensionless;

and: $V_2^{\text{cap}}(\rho = 1) = \frac{\boldsymbol{4D_{\text{a}}^2}}{\boldsymbol{\pi}} = 4\pi r^2 = A_2(r)$ 2-area of entire 2-sphere;

$\boldsymbol{n}$-surface ("circumference"):

$$\boldsymbol{A_n^{\text{cap}} = \frac{dV_n^{\text{cap}}}{dr_n} = \frac{1}{D_{\text{a}}} \cdot \frac{dV_n^{\text{cap}}(\rho)}{d\rho}}$$

S_3^{cap} = (non-Euclidean) S_2 , "circumference" = 2-surface area:

$$V_3^{\text{cap}}(\rho) = D_{\text{a}}^3 \frac{2\pi\rho - \sin 2\pi\rho}{\pi^2} \quad \therefore \quad A_3^{\text{cap}}(\rho) = \frac{D_{\text{a}}^2}{\pi^2} \cdot \frac{d}{d\rho}(2\pi\rho - \sin 2\pi\rho)$$

3-spherical: $\boldsymbol{A_3^{\text{cap}}(\rho) = \frac{4D_{\text{a}}^2}{\pi} \sin^2 \pi\rho}$

Euclidean: $4\pi r_{\text{cap}}^2 = 4\pi D_{\text{a}}^2 \rho^2 = \frac{4D_{\text{a}}^2}{\pi} (\pi\rho)^2$

S_2^{cap} = (non-Euclidean) S_1 , circumference (of ice cap) = 1-surface area:

$$V_2^{\text{cap}}(\rho) = D_{\text{a}}^2 \frac{2(1 - \cos \pi\rho)}{\pi} \quad \therefore \quad A_2^{\text{cap}}(\rho) = \frac{2D_{\text{a}}}{\pi} \cdot \frac{d}{d\rho}(1 - \cos \pi\rho)$$

2-spherical: $\boldsymbol{A_2^{\text{cap}}(\rho) = 2D_{\text{a}} \sin \pi\rho}$

Euclidean: $2\pi r_{\text{cap}} = 2\pi D_{\text{a}}\rho = 2D_{\text{a}}(\pi\rho)$

References:

1. S. Li: **Concise Formulas for the Area and Volume of a Hyperspherical Cap**
   Asian Journal of Mathematics and Statistic 4 (1): 66-70, 2011
   ISSN 1994-5418 / DOI: 10.3923/ajms.2011.66.70
   https://docsdrive.com/pdfs/ansinet/ajms/2011/66-70.pdf
   https://scialert.net/fulltext/fulltextpdf.php?pdf=ansinet/ajms/2011/66-70.pdf

2. **A Method for Computing the Incomplete Beta Function Ratio**
   U. S. Naval Weapons Laboratory Dahlgren, Virginia
   https://apps.dtic.mil/sti/tr/pdf/AD0642495.pdf

3. https://en.wikipedia.org/wiki/Gamma_function
4. https://en.wikipedia.org/wiki/Beta_function
5. https://en.wikipedia.org/wiki/Beta_function#Incomplete_beta_function
6. https://en.wikipedia.org/wiki/Volume_of_an_n-ball
7. https://en.wikipedia.org/wiki/N-sphere
8. https://en.wikipedia.org/wiki/Spherical_cap
9. https://en.wikipedia.org/wiki/Double_factorial