http://henk-reints.nl/astro/HR-Newtonian-gravitational-lensing.pdf:

Johann Georg von Soldner (1801, "Ueber die Ablenkung eines Lichtstrals"):

$$\Delta\theta_\mathrm{N} = 2\arcsin\frac{GM}{GM+r_\mathrm{p}c^2}$$

With: $r_\mathrm{S} = \frac{2GM}{c^2}$ and: $\frac{r_\mathrm{any}}{r_\mathrm{S}} =: \rho_\mathrm{any}$

we obtain: $\vartheta_\mathrm{N} \coloneqq \frac{\Delta\theta_\mathrm{N}}{2} = \arcsin\frac{1}{1+2\rho_\mathrm{p}} \approx \frac{1}{1+2\rho_\mathrm{p}}$ $\therefore \Delta\theta_\mathrm{N} \approx \frac{1}{\rho_\mathrm{p}}$

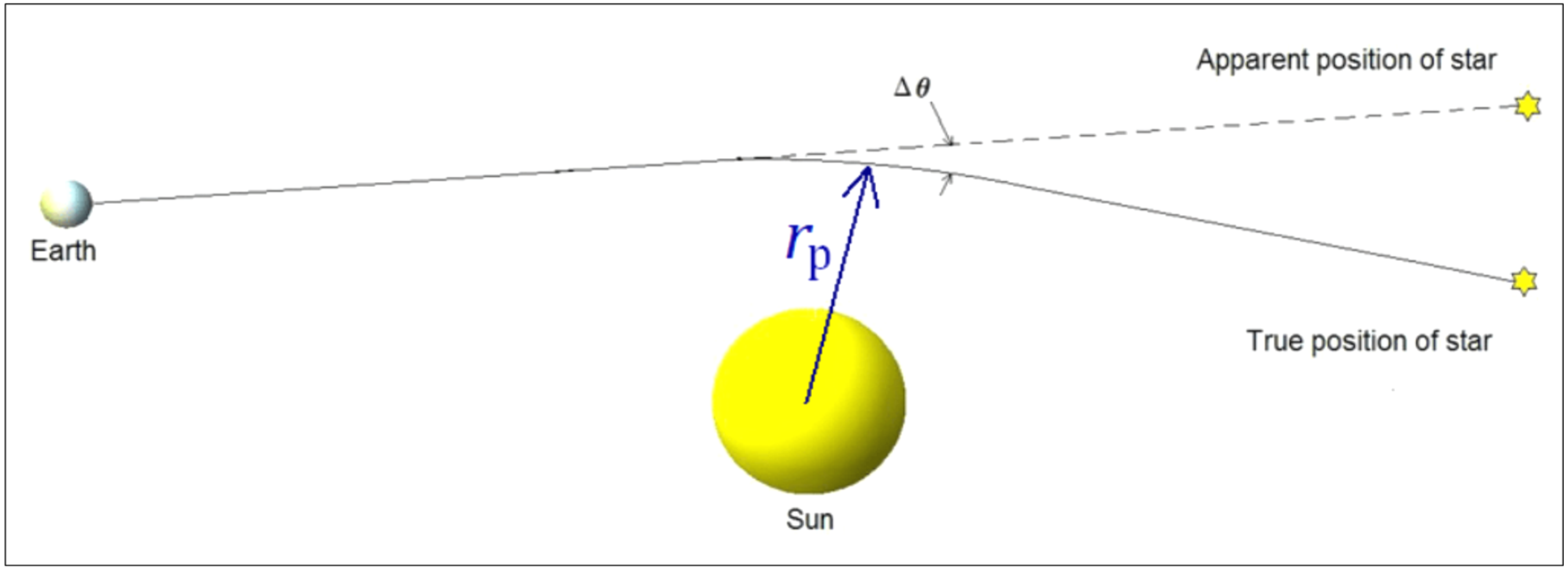
we'll later need: $\tan\vartheta_\mathrm{N} = \frac{1}{\sqrt{4\rho_\mathrm{p}^2+4\rho_\mathrm{p}}}$ $\left(\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}},\ x \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right)$

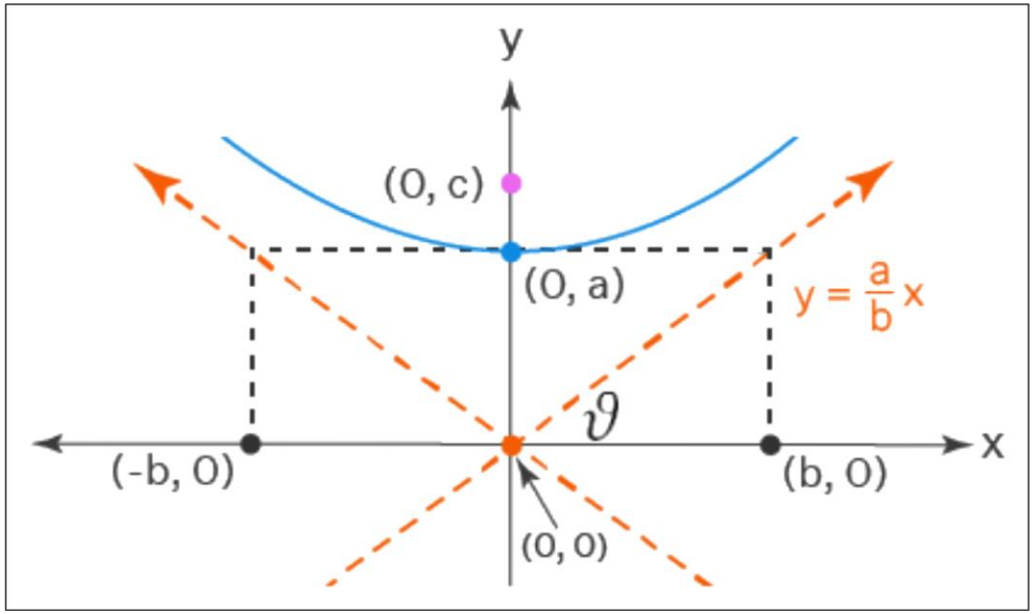
Einstein found: $\Delta\theta_\mathrm{E} \approx \frac{2r_\mathrm{S}}{r_\mathrm{p}} = \frac{2}{\rho_\mathrm{p}} \approx 2\Delta\theta_\mathrm{N}$

hence: $\vartheta_\mathrm{E} = 2\vartheta_\mathrm{N} \approx \frac{2}{1+2\rho_\mathrm{p}} \approx \arcsin\frac{2}{1+2\rho_\mathrm{p}}$ (presumably)

and: $\tan\vartheta_\mathrm{E} \approx \frac{2}{(1+2\rho_\mathrm{p})\sqrt{1-\left(\frac{2}{1+2\rho_\mathrm{p}}\right)^2}} = \frac{2}{\sqrt{(1+2\rho_\mathrm{p})^2-4}} = \frac{2}{\sqrt{4\rho_\mathrm{p}^2+4\rho_\mathrm{p}-3}}$

For the impatient: final explanation of factor 2 starts @page **31** .





$$a^2 + b^2 = c^2$$

$$r_\mathrm{p} = c - a$$

$$\frac{y}{a} = \sqrt{1 + \left(\frac{x}{b}\right)^2}$$

$$\vartheta = \vartheta_\mathrm{N}$$

Hyperbola: $y = a \cdot \sqrt{1 + x^2/b^2}$ (Newtonian trajectory)

we have: $\frac{a}{b} = \tan\vartheta_{\mathrm{N}} = \frac{1}{\sqrt{4\rho_{\mathrm{p}}^2 + 4\rho_{\mathrm{p}}}}$

hence: $b = a \cdot \sqrt{4\rho_{\mathrm{p}}^2 + 4\rho_{\mathrm{p}}}$ $\therefore b^2 = a^2\left(4\rho_{\mathrm{p}}^2 + 4\rho_{\mathrm{p}}\right)$

(not speed of light!): $c^2 = a^2 + b^2 = a^2\left(4\rho_{\mathrm{p}}^2 + 4\rho_{\mathrm{p}} + 1\right) = a^2\left(2\rho_{\mathrm{p}} + 1\right)^2$

Excentricity: $e = \frac{c}{a} = 2\rho_{\mathrm{p}} + 1$

Also: $c = r_{\mathrm{p}} + a = \rho_{\mathrm{p}} r_{\mathrm{S}} + a$

so: $c^2 = 4\rho_{\mathrm{p}}^2 a^2 + 4\rho_{\mathrm{p}} a^2 + a^2 = \rho_{\mathrm{p}}^2 r_{\mathrm{S}}^2 + 2a\rho_{\mathrm{p}} r_{\mathrm{S}} + a^2$

hence: $4\left(\rho_{\mathrm{p}}^2 + \rho_{\mathrm{p}}\right)a^2 - 2\rho_{\mathrm{p}} r_{\mathrm{S}} a - \rho_{\mathrm{p}}^2 r_{\mathrm{S}}^2 = 0$

therefore:
$$a = \frac{2\rho_\mathrm{p} r_\mathrm{S} \pm \sqrt{4\rho_\mathrm{p}^2 r_\mathrm{S}^2 - 4\cdot 4(\rho_\mathrm{p}^2 + \rho_\mathrm{p})(-\rho_\mathrm{p}^2 r_\mathrm{S}^2)}}{2(4\rho_\mathrm{p}^2 + 4\rho_\mathrm{p})}$$

hence:
$$a = \frac{2\rho_\mathrm{p} r_\mathrm{S} \pm 2\rho_\mathrm{p} r_\mathrm{S}\sqrt{1 + 4(\rho_\mathrm{p}^2 + \rho_\mathrm{p})}}{2(4\rho_\mathrm{p}^2 + 4\rho_\mathrm{p})} = r_\mathrm{S}\frac{1 \pm \sqrt{4\rho_\mathrm{p}^2 + 4\rho_\mathrm{p} + 1}}{4\rho_\mathrm{p} + 4}$$

or:
$$\frac{a}{r_\mathrm{S}} = \frac{1 + 2\rho_\mathrm{p} + 1}{4\rho_\mathrm{p} + 4} = \frac{1}{2}$$
(using "+" since $a > 0$)

Interesting: distance to other branch of hyperbola equals r_S .

Radius of curvature in periapsis:

$$r_\mathrm{curve,N} = \frac{b^2}{a} = 4a(\rho_\mathrm{p}^2 + \rho_\mathrm{p}) = 4\frac{r_\mathrm{S}}{2}(\rho_\mathrm{p}^2 + \rho_\mathrm{p})$$

$$\rho_\mathrm{curve,N} := \frac{r_\mathrm{curve,N}}{r_\mathrm{S}} = 2\rho_\mathrm{p}(\rho_\mathrm{p} + 1) = \rho_\mathrm{p}(1 + e)$$

$$r_\mathrm{curve,N} = 2r_\mathrm{p}\left(\frac{r_\mathrm{p}}{r_\mathrm{S}} + 1\right) = r_\mathrm{p}(1 + e)$$

Einsteinian: **PRESUMING** **a hyperbola** with $\vartheta = \vartheta_E$ instead of ϑ_N .

This time: $\frac{a}{b} = \tan \vartheta_E = \frac{2}{\sqrt{4\rho_p^2+4\rho_p-3}}$

hence: $b = \frac{a}{2}\sqrt{4\rho_p^2 + 4\rho_p - 3}$ $\quad \therefore b^2 = a^2\left(\rho_p^2 + \rho_p - \frac{3}{4}\right)$

(not speed of light!): $c^2 = a^2 + b^2 = a^2\left(\rho_p^2 + \rho_p + \frac{1}{4}\right) = a^2\left(\frac{2\rho_p+1}{2}\right)^2$

Excentricity: $e = \frac{c}{a} = \frac{2\rho_p+1}{2} = \rho_p + \frac{1}{2}$

Also: $c = r_p + a = \rho_p r_S + a$

so: $c^2 = a^2\left(\rho_p^2 + \rho_p + \frac{1}{4}\right) = \rho_p^2 r_S^2 + 2a\rho_p r_S + a^2$

hence: $\left(\rho_p^2 + \rho_p - \frac{3}{4}\right)a^2 - 2\rho_p r_S a - \rho_p^2 r_S^2 = 0$

therefore: $a = \frac{2\rho_p r_S \pm \sqrt{4\rho_p^2 r_S^2 - 4\left(\rho_p^2+\rho_p-\frac{3}{4}\right)(-\rho_p^2 r_S^2)}}{2\left(\rho_p^2+\rho_p-\frac{3}{4}\right)}$

hence: $$a = \frac{2\rho_\mathrm{p} r_\mathrm{S} \pm 2\rho_\mathrm{p} r_\mathrm{S}\sqrt{1+\left(\rho_\mathrm{p}^2+\rho_\mathrm{p}-\frac{3}{4}\right)}}{2\left(\rho_\mathrm{p}^2+\rho_\mathrm{p}-\frac{3}{4}\right)} = r_\mathrm{S}\frac{\rho_\mathrm{p} \pm \rho_\mathrm{p}\sqrt{\rho_\mathrm{p}^2+\rho_\mathrm{p}+\frac{1}{4}}}{\rho_\mathrm{p}^2+\rho_\mathrm{p}-\frac{3}{4}}$$

or: $$\frac{a}{r_\mathrm{S}} = \frac{\rho_\mathrm{p} + \rho_\mathrm{p}\sqrt{\rho_\mathrm{p}^2+\rho_\mathrm{p}+\frac{1}{4}}}{\rho_\mathrm{p}^2+\rho_\mathrm{p}-\frac{3}{4}} = \frac{2\rho_\mathrm{p}}{2\rho_\mathrm{p}-1}$$

Interesting: $\lim_{\rho_\mathrm{p}\to\infty}\frac{a}{r_\mathrm{S}} = 1$ trajectory no closer to directrix than r_S .

Radius of curvature in periapsis:

$$r_\mathrm{curve,E} = \frac{b^2}{a} = a\left(\rho_\mathrm{p}^2+\rho_\mathrm{p}-\frac{3}{4}\right) = \frac{2\rho_\mathrm{p} r_\mathrm{S}}{2\rho_\mathrm{p}-1}\left(\rho_\mathrm{p}^2+\rho_\mathrm{p}-\frac{3}{4}\right)$$

$$\rho_\mathrm{curve,E} := \frac{r_\mathrm{curve,E}}{r_\mathrm{S}} = \frac{4\rho_\mathrm{p}^3+4\rho_\mathrm{p}^2-3\rho_\mathrm{p}}{4\rho_\mathrm{p}-2} = \rho_\mathrm{p}\left(\rho_\mathrm{p}+\frac{3}{2}\right) = \rho_\mathrm{p}(1+e)$$

$$r_\mathrm{curve,E} = r_\mathrm{p}\left(\frac{r_\mathrm{p}}{r_\mathrm{S}}+\frac{3}{2}\right)$$

Altogether:

Newtonian:	Einsteinian:
$\Delta\theta = 2\arcsin\frac{1}{1+2\rho_p} \approx \frac{1}{\rho_p}$	$\Delta\theta \approx \frac{2}{\rho_p} \approx 2\Delta\theta_N \approx 2\arcsin\frac{2}{1+2\rho_p}$
$e = 2\rho_p + 1$	$e = \rho_p + \frac{1}{2}$
$\rho_{curve} = 2\rho_p(\rho_p + 1) = \rho_p(1+e)$	$\rho_{curve} = \rho_p\left(\rho_p + \frac{3}{2}\right) = \rho_p(1+e)$
$r_{curve} = 2r_p\left(\frac{r_p}{r_S} + 1\right) = r_p(1+e)$	$r_{curve} = r_p\left(\frac{r_p}{r_S} + \frac{3}{2}\right) = r_p(1+e)$
$\frac{a}{r_S} = \frac{1}{2}$	$\frac{a}{r_S} = \frac{2\rho_p}{2\rho_p - 1}$

		Newtonian			Einsteinian		
where	ρ_p	$\Delta\theta$	ρ_{curv}	e	$\Delta\theta$	ρ_{curv}	e
photon sphere	3/2	29°	7.5	4	60°	4.5	2
MBO	2	23°	12	5	47°	7	2.5
ISCO	3	16.5°	24	7	33°	13.5	3.5
S4714 @ Sgr A*	155	22′	4.8×10^4	311	44′	2.4×10^4	155
skim sun	2.36×10^5	**0.87"**	1.1×10^{11}	4.7×10^5	**1.75"**	5.6×10^{10}	2.36×10^5

I dare to PRESUME:

total deflection =
Newtonian hyperbola plus
relativistic perihelion shift thereof:

$$\Delta\boldsymbol{\theta}_{\mathbf{E}} = \Delta\boldsymbol{\theta}_{\mathbf{N}} + \Delta\boldsymbol{\varphi}_{\mathbf{hyp}}$$

I do not clearly see how to work this out, but why should I not try some 🎻fiddling🎶 & 🔨tinkering🔧 to get some idea?

Please note: *this will* ***not*** *yield* ***any proof***,

but it might give some (physical) insight.

In http://henk-reints.nl/astro/HR-Mercury-perihelion-precession-by-SR-only.pdf

I found the ~~perihelion~~ ***orbit*** precession

per orbit as: $$\Delta\varphi_{/\mathrm{orb}} = \boldsymbol{\frac{3\pi}{\rho}} - \frac{\pi}{\rho^2}$$

where: $\rho = \rho_\mathrm{p}(1+e) = \alpha(1-e^2) =$ *semi latus rectum*

and: $\alpha = a/r_\mathrm{S} =$ *semi major axis*

Einstein found: $$\Delta\varphi_{/\mathrm{orb}} = \frac{24\pi^3 a^2}{T^2 c^2 (1-e^2)} = \boldsymbol{\frac{3\pi}{\rho}}$$

which equals: $$\Delta\varphi_{/\mathrm{orb}} = \frac{3\pi}{\alpha(1-e^2)} = \frac{3\pi}{\rho_\mathrm{p}(1+e)}$$

PRESUMPTION: $$\Delta\varphi_\mathrm{hyp} = \frac{3\pi}{\alpha(e^2-1)} = \frac{3\pi}{\rho_\mathrm{p}(1+e)}$$

hence: $$\Delta\varphi_\mathrm{hyp} = \frac{3\pi}{\rho_\mathrm{p}(1+e)} = \Delta\varphi_{/\mathrm{orb}}$$ in 1st order.

Applying $\Delta\varphi_{\mathrm{hyp}} = \dfrac{3\pi}{\rho_{\mathrm{p}}(1+e)}$ to the sun:

Newtonian: $e = 2\rho_{\mathrm{p}} + 1$

$$\Delta\varphi_{\mathrm{hyp}} = \frac{3\pi}{2\rho_{\mathrm{p}}(\rho_{\mathrm{p}}+1)}$$

https://www.iau.org/static/resolutions/IAU2015_English.pdf :

$r_{\odot} \approx 6.957 \times 10^{8}$ m

$r_{\mathrm{S},\odot} \approx 2953$ m

so: $\rho_{\odot} \approx 2.356 \times 10^{5}$

yielding: $\Delta\varphi \approx 8.49 \times 10^{-11}$ rad

$\approx 1.75 \times 10^{-5}$ arcsec

WAY TOO SMALL! 👎☹

Isn't trajectory's curvature k a measure of gravity?

Newton: $g = \frac{GM}{r^2} = GM \cdot k^2$

$$v_{\text{esc,orb}} \propto \sqrt{1/r} = \sqrt{k}$$

$$\boldsymbol{E}_{\textbf{esc,orb}} \propto \boldsymbol{v}^{\mathbf{2}}_{\textbf{esc,orb}} \propto \boldsymbol{k}$$

PRESUME: curvature affects energy.

& we'll keep it as easy as possible,
i.e. no exponent for k .

We'll choose a section of trajectory around periapsis, having angle 2ψ as seen from focus. Then we'll try $\Delta\varphi_{\text{hyp}} = \frac{2\psi}{2\pi}\Delta\varphi_{/\text{orb}}k^*_{\text{avg}}$, where $k^*_{\text{avg}} =$ estimated overall average curvature (as fraction of max. curv.), for which we'll use the curvature at the section's end points. For *some* section it might (should?) yield $\Delta\varphi_{\text{hyp}} = \Delta\theta_{\text{E}} - \Delta\theta_{\text{N}} \approx 1/\rho_{\text{p}}$.

First, we make the Newtonian hyperbolic trajectory dimensionless:

$$y = a\sqrt{\frac{x^2}{b^2} + 1}$$

we have: $a = \frac{r_S}{2}$ & $b^2 = 4a^2(\rho_p^2 + \rho_p) = r_S^2(\rho_p^2 + \rho_p)$

we'll later need: $c = r_p + a = r_S\left(\rho_p + \frac{1}{2}\right) \therefore \frac{c}{r_S} = \rho_p + \frac{1}{2}$

trajectory: $\frac{y}{r_S} = \frac{1}{2}\sqrt{\frac{x^2}{r_S^2(\rho_p^2+\rho_p)} + 1}$

we define: $\boldsymbol{\xi} \coloneqq \frac{\boldsymbol{x}}{\boldsymbol{r_S}}$ and: $\boldsymbol{\eta} \coloneqq \frac{\boldsymbol{y}}{\boldsymbol{r_S}}$

yielding the dimensionless trajectory: $\eta = \frac{1}{2}\sqrt{\frac{\xi^2}{\rho_p^2+\rho_p} + 1} = \sqrt{\frac{\xi^2+\rho_p^2+\rho_p}{4(\rho_p^2+\rho_p)}}$

we also define: $\boldsymbol{R^2 = \rho_p(\rho_p + 1)}$ (hence $R \approx \rho_p$ if $\rho_p \gg 1$ and $R^2 \geq 2$ since $\rho_p \geq 1$)

yielding: $\boldsymbol{\eta} = \sqrt{\frac{\xi^2+R^2}{4R^2}} = \frac{\boldsymbol{\sqrt{\xi^2+R^2}}}{\boldsymbol{2R}}$

Curvature of $y = f(x)$: $$k(x) = \frac{f''(x)}{(1+[f'(x)]^2)^{3/2}}$$

hyperbolic trajectory: $$\eta = \frac{\sqrt{\xi^2+R^2}}{2R}$$

$$\frac{d\eta}{d\xi} = \frac{\xi}{2R\sqrt{\xi^2+R^2}} \qquad \left(\frac{d\eta}{d\xi}\right)^2 = \frac{\xi^2}{4R^2(\xi^2+R^2)}$$

$$\frac{d^2\eta}{d\xi^2} = \frac{R}{2(\xi^2+R^2)^{3/2}}$$

so: $$k(\xi) = \frac{\frac{R}{2(\xi^2+R^2)^{3/2}}}{\left(1+\frac{\xi^2}{4R^2(\xi^2+R^2)}\right)^{3/2}} = \frac{R}{2(\xi^2+R^2)^{3/2}\left(\frac{4R^2(\xi^2+R^2)+\xi^2}{4R^2(\xi^2+R^2)}\right)^{3/2}}$$

$$= \frac{R}{2\left((\xi^2+R^2)\frac{4R^2(\xi^2+R^2)+\xi^2}{4R^2(\xi^2+R^2)}\right)^{3/2}} = \frac{R}{2\left(\frac{4R^2(\xi^2+R^2)+\xi^2}{4R^2}\right)^{3/2}} = \frac{R}{2\frac{(4R^2(\xi^2+R^2)+\xi^2)^{3/2}}{(4R^2)^{3/2}}}$$

$$= \frac{R\cdot(4R^2)^{3/2}}{2(4R^2(\xi^2+R^2)+\xi^2)^{3/2}} = \frac{R\cdot(4)^{3/2}(R^2)^{3/2}}{2(4R^2(\xi^2+R^2)+\xi^2)^{3/2}} = \frac{8R^4}{2(4R^2(\xi^2+R^2)+\xi^2)^{3/2}}$$

which yields: $$\boldsymbol{k(\xi)} = \frac{4R^4}{(4R^2(\xi^2+R^2)+\xi^2)^{3/2}} = \boldsymbol{\frac{4R^4}{(4R^4+(4R^2+1)\xi^2)^{3/2}}}$$

maximum at $\xi = 0$: $$k_{\max} = \frac{4R^4}{(4R^2(0^2+R^2)+0^2)^{3/2}} = \frac{4R^4}{(4R^4)^{3/2}} = \frac{1}{2R^2}$$

relative curvature: $$k^*(\xi) := \frac{k(\xi)}{k_{\max}} = \frac{8R^6}{(4R^4+(4R^2+1)\xi^2)^{3/2}}$$

hence: $$k^*(\xi) \in (0,1]$$

We'll also need: $$\tan\psi_{\text{avg}} = \frac{\xi_{\text{avg}}}{\frac{c}{r_S}-\eta_{\text{avg}}} = \frac{\xi_{\text{avg}}}{\frac{1}{2}+\rho_p-\eta_{\text{avg}}} = \frac{\xi_{\text{avg}}}{\frac{1}{2}+\rho_p-\frac{\sqrt{\xi_{\text{avg}}^2+R^2}}{2R}}$$

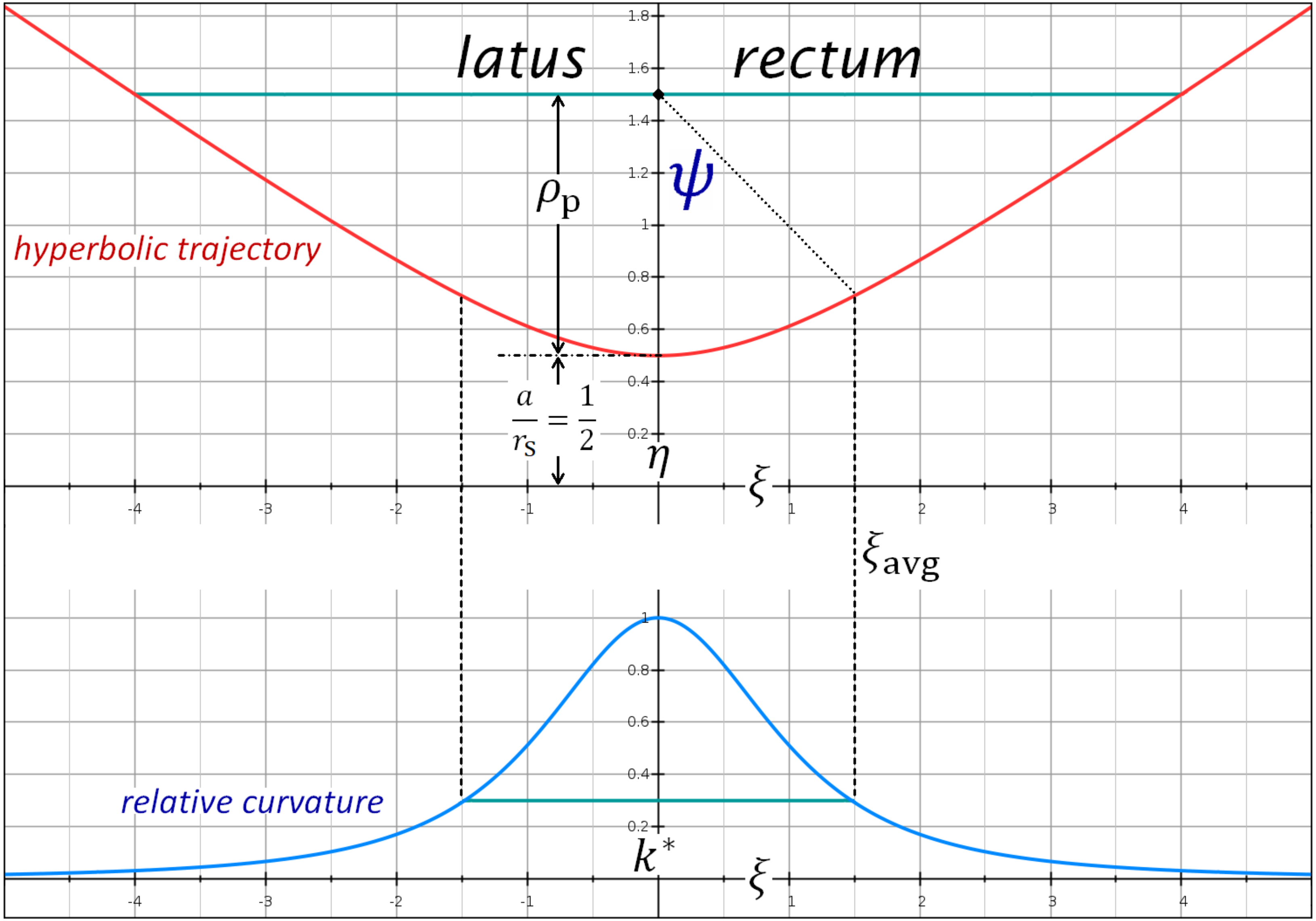
$$= \frac{2R\xi_{\text{avg}}}{R(2\rho_p+1)-\sqrt{\xi_{\text{avg}}^2+R^2}}$$

hence: $$\psi_{\text{avg}} = \arctan\frac{2R\xi_{\text{avg}}}{R(2\rho_p+1)-\sqrt{\xi_{\text{avg}}^2+R^2}}$$

WANTED: $$\xi_{\text{avg}} \Rightarrow k^*_{\text{avg}}\frac{2\psi_{\text{avg}}}{2\pi}\left(\frac{3\pi}{\rho_p}-\frac{\pi}{\rho_p^2}\right) \approx \frac{1}{\rho_p}$$

i.e. $$k^*_{\text{avg}}\psi_{\text{avg}}\left(\frac{3}{\rho_p}-\frac{1}{\rho_p^2}\right) \approx \frac{1}{\rho_p}$$ or: $$3k^*_{\text{avg}}\psi_{\text{avg}} \approx 1$$

Example $(\rho_{\mathrm{p}} = 1 \therefore R^2 = 2)$: $\eta = \sqrt{\frac{\xi^2+2}{8}}$ & $k^*(\xi) = \frac{64}{(9\xi^2+16)^{3/2}}$



We need: k^{*inv}: $\xi(k^*) = \pm 2R^2\sqrt{\frac{(k^*)^{-2/3}-1}{4R^2+1}}$

We will try ξ_{avg} corresponding to:

A) latus rectum: $\xi_{\text{lr}} = 2\rho_{\text{p}}(\rho_{\text{p}}+1)$ (= ρ_{curve} at periapsis);

B) FWHM of curvature: $\xi_{\text{hm}} = k^{*\text{inv}}\left(k^*_{\text{hm}} = \frac{1}{2}\right) = 2R^2\sqrt{\frac{\sqrt[3]{4}-1}{4R^2+1}}$;

C) inflection points of curv.: $\xi_{\text{flex}} = \frac{R^2}{\sqrt{4R^2+1}}$ (see appendix 1);

D) std. dev. of curvature: $\xi_\sigma = \sqrt{\frac{\int_0^\infty \xi^2 k^*(\xi)d\xi}{\int_0^\infty k^*(\xi)d\xi}}$ (see appendix 2);

E) mean curvature over: $\alpha = \arctan\frac{d\eta}{d\xi}$: $\xi_\mu = k^{*\text{inv}}\left(k^*_\mu = \frac{\int_0^\vartheta k^*(\alpha)d\alpha}{\vartheta}\right)$

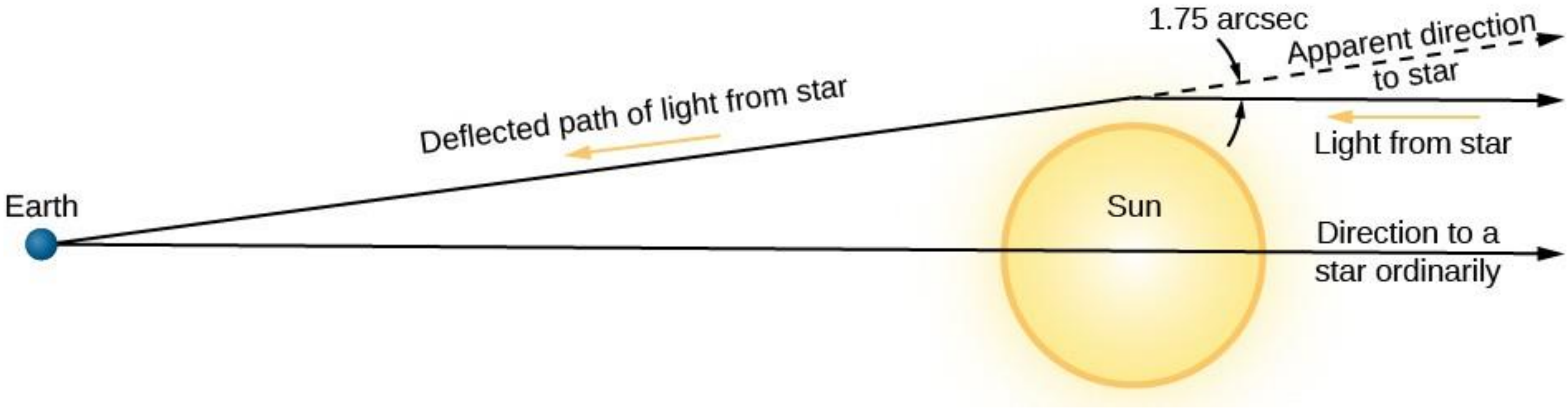
(needs conversion from $k^*(\xi)$ to $k^*(\alpha)$) (see appendix 3).

Hopefully, we find an acceptable one.

For light skimming the sun, we have:

https://www.iau.org/static/resolutions/IAU2015_English.pdf:

	$r_{\odot}$	$\approx 6.957 \times 10^8$	m
	$r_{S,\odot}$	≈ 2953	m
so:	$\rho_{\odot}$	$\approx 2.356 \times 10^5$	
and:	R^2	$\approx 5.549 \times 10^{10}$	



https://courses.lumenlearning.com/suny-geneseo-astronomy/chapter/tests-of-general-relativity/

Latus rectum: $\xi_{\mathrm{lr}} = 2\rho_{\mathrm{p}}\left(\rho_{\mathrm{p}} + 1\right) \approx 1.11 \times 10^{11}$

we find: $k_{\mathrm{lr}}^{*} = \frac{8R^6}{\left(4R^4+(4R^2+1)\xi_{\mathrm{lr}}^2\right)^{3/2}} \approx 9.56 \times 10^{-18}$

and obviously: $\psi_{\mathrm{lr}} = \pi/2$

∴ $\mathbf{3}\boldsymbol{k_{\mathrm{lr}}^{*}\psi_{\mathrm{lr}}} \approx \mathbf{4.51 \times 10^{-17}}$ 👎👎😟

For a ray of light, the *latus rectum* defines a way too long ($l \approx 4382$ au) nearly straight section with for the very most part $r \ggg r_{\mathrm{S}} \approx 2953$ m, so the overall *curvature* (i.e. *gravitational effect*) vanishes.

***Latus rectum* does *not* do the job.** 😥

Full Width at Half Maximum: $\xi_{\mathrm{hm}} = 2R^2\sqrt{\frac{\left(\sqrt[3]{4}-1\right)}{(4R^2+1)}} \approx 1.81 \times 10^5$

by definition: $k^*_{\mathrm{hm}} = \frac{1}{2}$

and: $\psi_{\mathrm{hm}} = \arctan\frac{2R\xi_{\mathrm{hm}}}{R(2\rho_{\mathrm{p}}+1)-\sqrt{\xi^2_{\mathrm{hm}}+R^2}} \approx 0.208\cdot\pi$

∴ $\mathbf{3k^*_{hm}\psi_{hm} \approx 0.981 \approx 1}$ ☺👍

FWHM seems a fairly good approximation!

Inflection points: $\xi_{\text{flex}} = R^2/\sqrt{4R^2+1} \approx 1.18 \times 10^5$

We find: $$k^*_{\text{flex}} = \frac{8R^6}{\left(4R^4+(4R^2+1)\xi^2_{\text{flex}}\right)^{3/2}} = \frac{8}{\sqrt{125}}$$

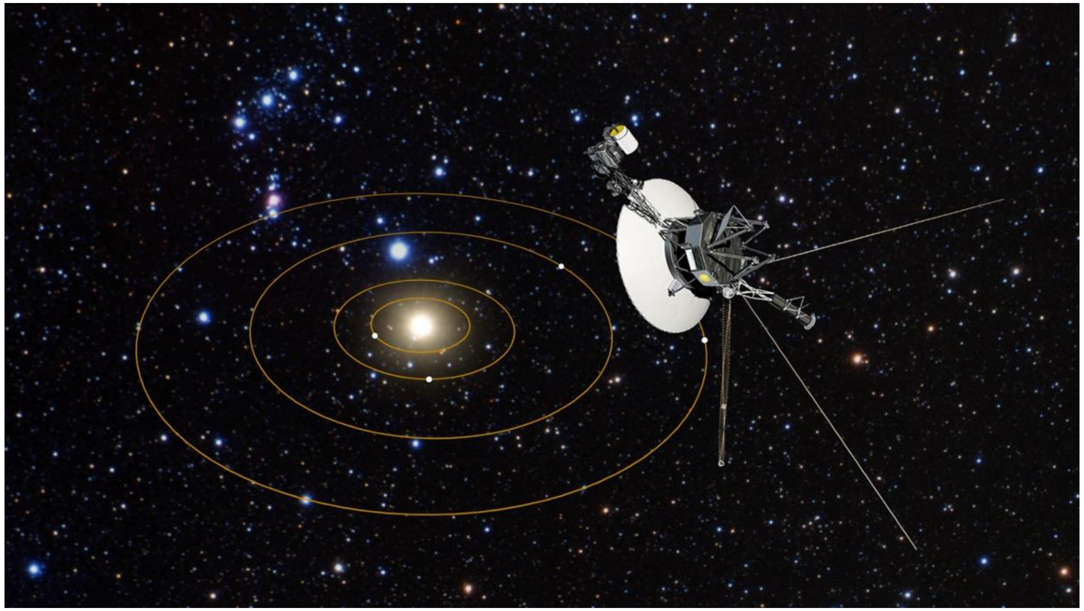
and: $$\psi_{\text{flex}} = \arctan\frac{2R\xi_{\text{flex}}}{R(2\rho_{\text{p}}+1)-\sqrt{\xi^2_{\text{flex}}+R^2}} \approx 0.148\cdot\pi$$

∴ $$\mathbf{3k^*_{\text{flex}}\psi_{\text{flex}} \approx 0.995 \approx 1}$$ 😀👍👍

Flex points yield a much better approximation!

Standard deviation: NOPE.

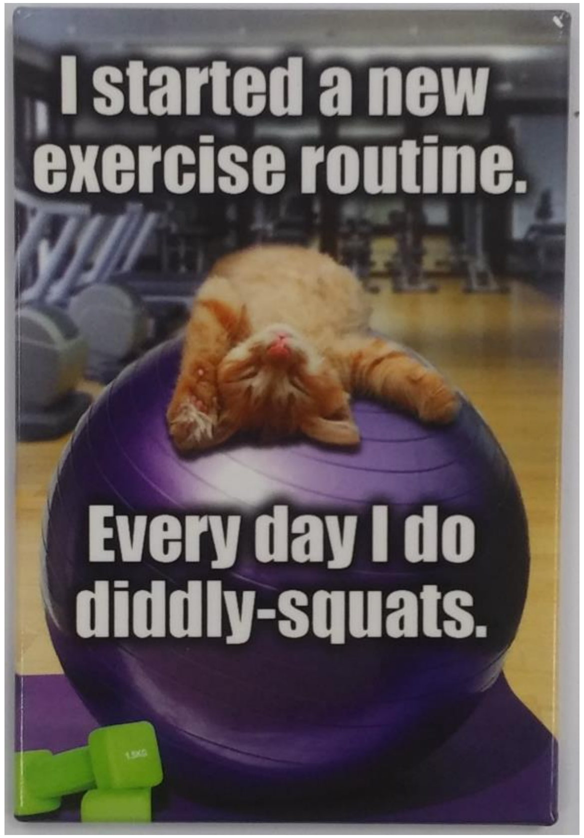
See appendix 2: the integral does not converge.



All the way to infinity...

Mean curvature over total deflection: NEITHER.

See appendix 3:
it yields a zero size section of the trajectory.



It seems the inflection points of the curvature yield the best approximation. It also *feels* like a good & elegant way to find the trajectory's perihelion shift.

Derivation of general flex-based equation:

$$R^2 = \rho_\mathrm{p}(\rho_\mathrm{p} + 1)$$

$$\xi_\mathrm{flex} = R^2/\sqrt{4R^2+1}$$

$$k^*_\mathrm{flex} = \frac{8R^6}{\left(4R^4+(4R^2+1)\xi^2_\mathrm{flex}\right)^{3/2}} = \frac{8R^6}{\left(4R^4+(4R^2+1)\frac{R^4}{4R^2+1}\right)^{3/2}} = \frac{8R^6}{(5R^4)^{3/2}} = \frac{8}{\sqrt{125}}$$

$$\psi_\mathrm{flex} = \arctan\frac{2R\xi_\mathrm{flex}}{R(2\rho_\mathrm{p}+1)-\sqrt{\xi^2_\mathrm{flex}+R^2}} = \arctan\frac{2R^2R^2/\sqrt{4R^2+1}}{R^2(2\rho_\mathrm{p}+1)-\sqrt{R^2\frac{R^4}{4R^2+1}+R^4}}$$

$$= \arctan\frac{2R^4}{R^2(2\rho_\mathrm{p}+1)\sqrt{4R^2+1}-\sqrt{\frac{R^6(4R^2+1)}{4R^2+1}+R^4(4R^2+1)}} = \arctan\frac{2R^4}{R^2(2\rho_\mathrm{p}+1)\sqrt{4R^2+1}-\sqrt{5R^6+R^4}}$$

$$= \arctan \frac{2R^2}{(2\rho_p+1)\sqrt{4R^2+1}-\sqrt{5R^2+1}} = \arctan \frac{2\rho_p(\rho_p+1)}{(2\rho_p+1)\sqrt{4\rho_p(\rho_p+1)+1}-\sqrt{5\rho_p(\rho_p+1)+1}}$$

$$= \arctan \frac{2\rho_p(\rho_p+1)}{(2\rho_p+1)\sqrt{(2\rho_p+1)^2}-\sqrt{5\rho_p(\rho_p+1)+1}} = \arctan \frac{2\rho_p(\rho_p+1)}{(2\rho_p+1)^2-\sqrt{5\rho_p(\rho_p+1)+1}}$$

Laurent series at $\rho_p = \infty$: $\arctan\frac{1}{2} + \frac{1}{2\rho_p\sqrt{5}} + \mathcal{O}\left(\frac{1}{\rho_p^2}\right)$

$$\Delta\varphi_{hyp} = k^*_{flex}\psi_{flex}\left(\frac{3}{\rho_p} - \frac{1}{\rho_p^2}\right) =$$

$$\frac{8}{\sqrt{125}}\left(\frac{3}{\rho_p} - \frac{1}{\rho_p^2}\right) \arctan \frac{2\rho_p(\rho_p+1)}{(2\rho_p+1)^2-\sqrt{5\rho_p(\rho_p+1)+1}}$$

Or, for $\rho_p \gg 1$:

$$\Delta\varphi_{hyp} \approx \frac{24}{\rho_p\sqrt{125}}\left(\arctan\frac{1}{2} + \frac{1}{2\rho_p\sqrt{5}}\right)$$

$$\boldsymbol{\Delta\varphi_{hyp}} \approx \frac{24\arctan\frac{1}{2}}{\rho_p\sqrt{125}} + \frac{12}{25\rho_p^2} \approx \frac{\mathbf{0.99528}}{\boldsymbol{\rho_p}}$$

We have now found:

$$\Delta\varphi_{\text{hyp}} = \frac{8}{\sqrt{125}}\left(\frac{3}{\rho_{\text{p}}} - \frac{1}{\rho_{\text{p}}^2}\right)\arctan\frac{2\rho_{\text{p}}(\rho_{\text{p}}+1)}{(2\rho_{\text{p}}+1)^2 - \sqrt{5\rho_{\text{p}}(\rho_{\text{p}}+1)+1}}$$

Please take good notice:

this quasi-exact equation

has *NOT* been *PROVEN* !

Not deduced from only certainties.

I hope however that the above approach has increased your (physical) insight.

$$\Delta\varphi_{\mathrm{hyp}} = \frac{8}{\sqrt{125}}\left(\frac{3}{\rho_{\mathrm{p}}} - \frac{1}{\rho_{\mathrm{p}}^2}\right) \arctan \frac{2\rho_{\mathrm{p}}(\rho_{\mathrm{p}}+1)}{(2\rho_{\mathrm{p}}+1)^2 - \sqrt{5\rho_{\mathrm{p}}(\rho_{\mathrm{p}}+1)+1}}, \quad \Delta\theta_{\mathrm{N}} = 2\arcsin\frac{1}{1+2\rho_{\mathrm{p}}}, \quad \Delta\theta_{\mathrm{E}} = \Delta\theta_{\mathrm{N}} + \Delta\varphi_{\mathrm{hyp}}$$

ρ_{p}	$\Delta\varphi_{\mathrm{hyp}}$	$\Delta\theta_{\mathrm{N}}$	$\Delta\theta_{\mathrm{E}}$
1	$\frac{16}{5\sqrt{5}}\arctan\frac{4}{9-\sqrt{11}} = 50°.286$	$2\arcsin\frac{1}{3} = 38°.942$	$= 89°.228$
3/2	$\frac{112}{45\sqrt{5}}\arctan\frac{15}{32-\sqrt{79}} = 36°.714$	$2\arcsin\frac{1}{4} = 28°.955$	$= 65°.669$
2	$\frac{2}{\sqrt{5}}\arctan\frac{12}{25-\sqrt{31}} = 28°.350$	$2\arcsin\frac{1}{5} = 23°.074$	$= 51°.424$
3	$\frac{64}{45\sqrt{5}}\arctan\frac{24}{49-\sqrt{61}} = 19°.226$	$2\arcsin\frac{1}{7} = 16°.426$	$= 35°.653$
4	$\frac{11}{10\sqrt{5}}\arctan\frac{40}{81-\sqrt{101}} = 14°.469$	$2\arcsin\frac{1}{9} = 12°.759$	$= 27°.228$
5	$\frac{112}{125\sqrt{5}}\arctan\frac{60}{121-\sqrt{151}} = 11°.578$	$2\arcsin\frac{1}{11} = 10°.432$	$= 22°.010$
10	$\frac{58}{125\sqrt{5}}\arctan\frac{220}{441-\sqrt{551}} = 5°.7656$	$2\arcsin\frac{1}{21} = 5°.4588$	$= 11°.224$
sun	$= 0''.8707$	$= 0''.8748$	$= 1''.7455$

Deflections may seem incorrect for the rather small values of ρ_{p} .

My approach was focused on E's result $\Delta\theta_{\mathrm{E}} \approx 2/\rho_{\mathrm{p}}$ (hence $\Delta\varphi_{\mathrm{hyp}} \approx 1/\rho_{\mathrm{p}}$),
which actually is only a 1st order approximation;
for large $\Delta\varphi$, the values of k^*_{flex} & ψ_{flex} become incorrect since
they are based on a hyperbolic trajectory which it no longer is.

Update 2023-05-25:

Maybe next yields the correct perihelion shift of the hyperbolic trajectory of a ray of light.

Radius of curvature at ξ : $\varrho(\xi) = 1/k(\xi)$

Shift corresponding to $\varrho(\xi)$: $\Delta\varphi_{/\mathrm{orb}}(\xi) = \frac{3\pi}{\varrho(\xi)} - \frac{\pi}{\varrho(\xi)^2} = \pi[3 - k(\xi)]k(\xi)$

Infinitesimal path length: $d\sigma = \sqrt{d\xi^2 + d\eta^2}$

Infinitesimal angle: $d\psi = \frac{d\sigma}{\varrho(\xi)} = k(\xi)d\sigma$

Fraction of full orbit: $\frac{d\psi}{2\pi} = \frac{k(\xi)d\sigma}{2\pi}$

Infinitesimal shift: $d\varphi = \Delta\varphi_{/\mathrm{orb}}\frac{d\psi}{2\pi} = \frac{\Delta\varphi_{/\mathrm{orb}}}{2\pi}k(\xi)d\sigma$

becomes: $d\varphi = \frac{1}{2}[3 - k(\xi)]k(\xi)k(\xi)d\sigma$

which should be integrated over the entire trajectory.

The result should approach $\Delta\theta_{\mathrm{N}} = 2\arcsin\frac{1}{1+2\rho_{\mathrm{p}}} \approx \frac{1}{\rho_{\mathrm{p}}}$.

We have: $R^2 = \rho_\mathrm{p}(\rho_\mathrm{p} + 1)$

and: $\eta = \frac{\sqrt{\xi^2+R^2}}{2R}$

as well as: $k(\xi) = \frac{4R^4}{(4R^4+(4R^2+1)\xi^2)^{3/2}}$

We derive the derivative: $\frac{d\eta}{d\xi} = \frac{\xi}{2R\sqrt{\xi^2+R^2}}$

hence: $d\eta = \frac{\xi d\xi}{2R\sqrt{\xi^2+R^2}} \therefore d\eta^2 = \frac{\xi^2 d\xi^2}{4R^2(\xi^2+R^2)}$

therefore: $d\sigma = \sqrt{d\xi^2 + d\eta^2} = \sqrt{1 + \frac{\xi^2}{4R^2(\xi^2+R^2)}}d\xi$

or: $d\sigma = \sqrt{\frac{4R^2+1}{4R^2} - \frac{1}{4R^2+4\xi^2}}d\xi$

yielding: $d\varphi = \frac{1}{2}[3 - k(\xi)]k(\xi)^2\sqrt{\frac{4R^2+1}{4R^2} - \frac{1}{4R^2+4\xi^2}}d\xi$

which expands to: $d\varphi = \frac{1}{2}\left(3 - \frac{4R^4}{(4R^4+(4R^2+1)\xi^2)^{3/2}}\right)\frac{16R^8}{(4R^4+(4R^2+1)\xi^2)^3}\sqrt{\frac{4R^2+1}{4R^2} - \frac{1}{4R^2+4\xi^2}}d\xi$

and we need: $\Delta\varphi = \frac{1}{2}\int_{-\infty}^{\infty}\left(3 - \frac{4R^4}{(4R^4+(4R^2+1)\xi^2)^{3/2}}\right)\frac{16R^8}{(4R^4+(4R^2+1)\xi^2)^3}\sqrt{\frac{4R^2+1}{4R^2} - \frac{1}{4R^2+4\xi^2}}d\xi$

YEAH...

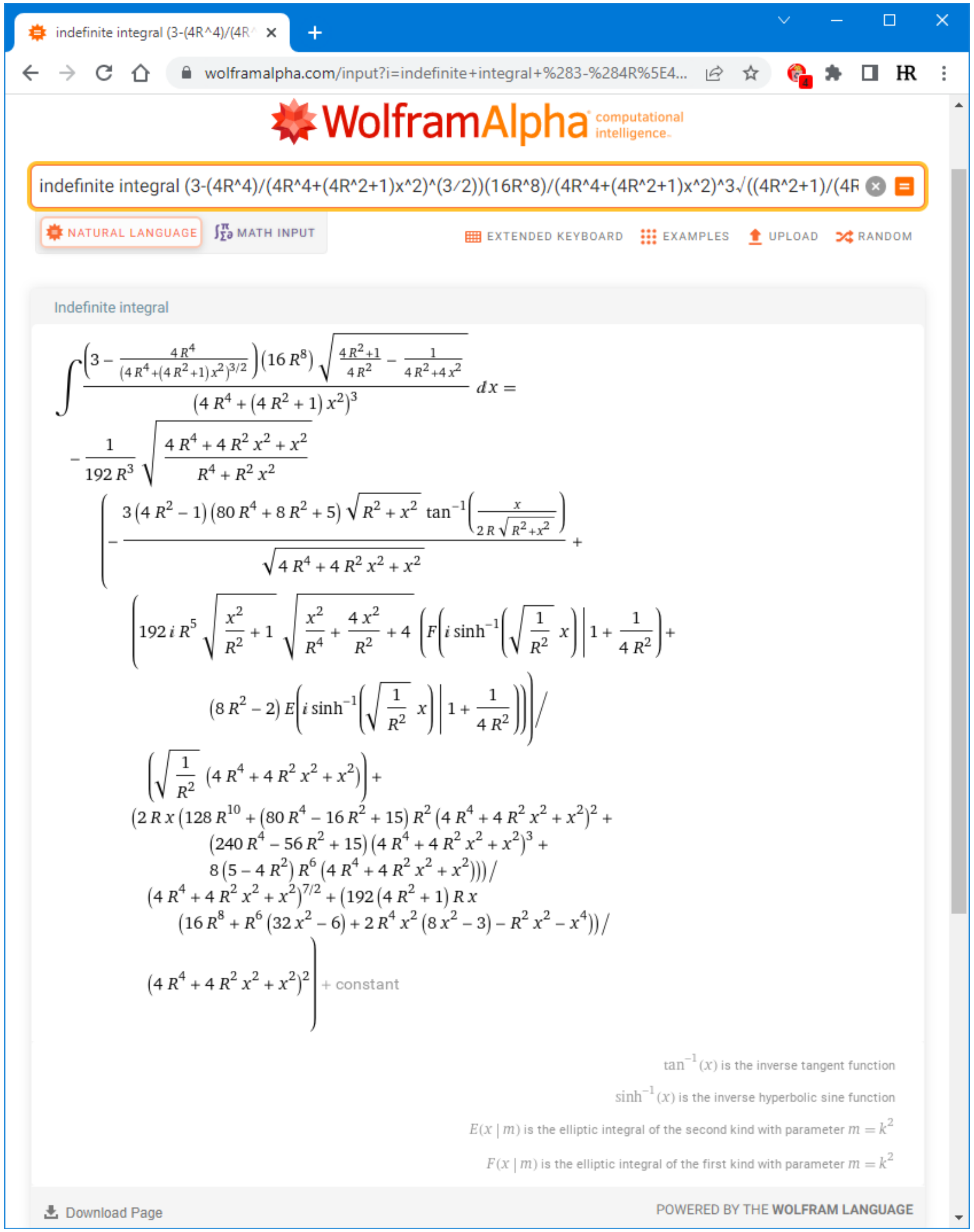
WolframAlpha *does* come up with the indefinite integral:

Nonetheless, I conquered the DRAGON numerically.

For light skimming the sun, it yields roughly:

$$\mathbf{0.7 \times 10^{-11}}\ \textbf{arcsec,}$$

which is WAY too less, so **this is not a correct method.**



Instead of: $$\Delta\varphi_{/\mathrm{orb}}(\xi) = \frac{3\pi}{\varrho(\xi)} - \frac{\pi}{\varrho(\xi)^2} = \pi[3 - k(\xi)]k(\xi)$$

I could/should have used: $$\frac{\Delta\varphi_{/\mathrm{orb,HR}}(\xi)}{2\pi} = \frac{6\varrho(\xi)+1}{(2\varrho(\xi)+1)^2} = \frac{(k(\xi)+6)k(\xi)}{(k(\xi)+2)^2}$$

(as derived in: http://henk-reints.nl/astro/HR-Mercury-perihelion-precession-by-SR-only.pdf)

yielding: $$d\varphi = \frac{1}{2}\cdot\frac{(k(\xi)+6)k(\xi)}{(k(\xi)+2)^2}k(\xi)\sqrt{\frac{4R^2+1}{4R^2} - \frac{1}{4R^2+4\xi^2}}d\xi$$

or: $$d\varphi = \frac{(k(\xi)+6)k(\xi)^2}{2(k(\xi)+2)^2}\sqrt{\frac{4R^2+1}{4R^2} - \frac{1}{4R^2+4\xi^2}}d\xi$$

I see yet another leviathan looming.

It yields the same: $\mathbf{0.7\times 10^{-11}}$ **arcsec,**

so **this is also not correct.**

Neither do numerical trials of various other methods,

e.g. using distance to focus instead of radius of curvature $@(\xi,\eta)$,

yield a proper deflection angle by integration.

http://henk-reints.nl/astro/HR-Mercury-perihelion-precession-by-SR-only.pdf

clarifies that ~~perihelion~~ orbit precession
has to do with the proper orbital period and
path length as perceived by the planet itself.

However, light perceives
ZERO travel distance in ZERO travel time,

so it may well be that the above presumption
of a trajectory's perihelion shift is flapdoodle.

Moreover, the orbital period of a hyperbola is
a meaningless concept, since it exceeds infinity.

BUT:

Wouldn't *sunlight*, emitted horizontally (as seen overthere)
by the sun's edge, undergo the ***very same*** deflection
as *starlight* coming from far beyond & skimming it?
Wouldn't that enlarge the sun's ***apparent*** radius
by the ***very same*** angle as the deflection?
Wouldn't the edge simply be blown outward by ***autolensing***?
Wouldn't it be that the edge may also be a fictitious one?
Wouldn't this apply to both incoming and outgoing light?
Wouldn't this apparently push the perihelion away by the ***same***
observation angle as the deflection of the ray of starlight?
Wouldn't this ***exactly*** render a factor of 2 as found by Einstein?
Wouldn't Soldner, precisely twice, suffice for thy wise nice eyes?

And wouldn't ***each*** *&* ***every*** *observed angular size be a blown-up one?*

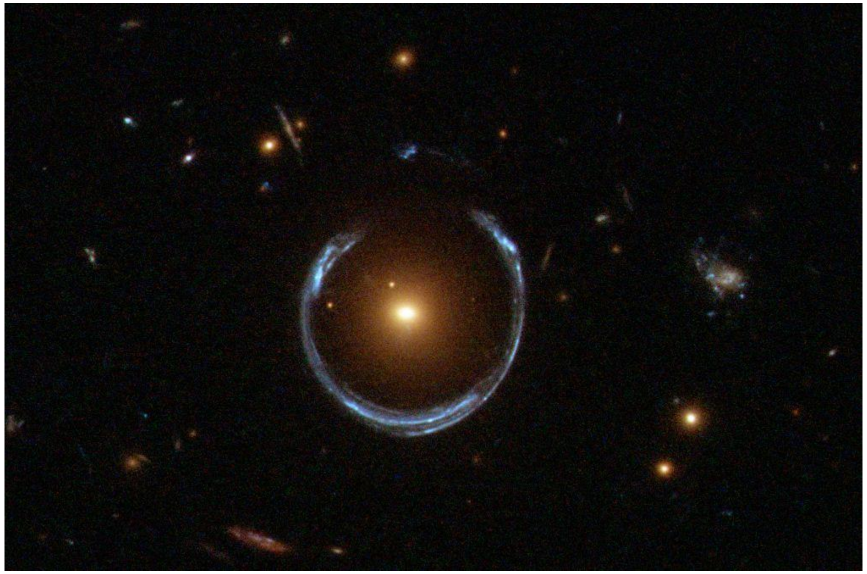
No need for the behemoth
of tensor calculus or whatever.

But the factor of 2 is ***not** really exact* and neither is the derivation below (among other things, ϑ_{N} applies to $\rho_{\mathrm{obs}} = \infty$), but it should give some insight.

	physical	apparent
spatial radius:	ρ_{phys}	ρ_{app}
observation distance:	ρ_{obs}	
angular radius:	$\phi_{\mathrm{phys}} = \arctan \frac{\rho_{\mathrm{phys}}}{\rho_{\mathrm{obs}}}$	$\phi_{\mathrm{app}} = \arctan \frac{\rho_{\mathrm{app}}}{\rho_{\mathrm{obs}}}$
yielding:	$\rho_{\mathrm{phys}} = \rho_{\mathrm{obs}} \tan \phi_{\mathrm{phys}}$	$\rho_{\mathrm{app}} = \rho_{\mathrm{obs}} \tan \phi_{\mathrm{app}}$
autolensing:	$\phi_{\mathrm{phys}} + \vartheta_{\mathrm{N}}(\rho_{\mathrm{phys}}) = \phi_{\mathrm{app}}$	
hence:	$\phi_{\mathrm{phys}} = \phi_{\mathrm{app}} - \vartheta_{\mathrm{N}}(\rho_{\mathrm{phys}})$	
so we obtain:	$\boldsymbol{\rho_{\mathrm{phys}}} = \boldsymbol{\rho_{\mathrm{obs}}} \tan\left(\boldsymbol{\phi_{\mathrm{app}}} - \arcsin \frac{1}{1+2\boldsymbol{\rho_{\mathrm{phys}}}}\right)$ $\boldsymbol{\rho_{\mathrm{obs}}}$ & $\boldsymbol{\phi_{\mathrm{app}}}$ are known, ***YOU*** find $\boldsymbol{\rho_{\mathrm{phys}}}$**!**[1]	
total unilateral deflection:	$\vartheta_{\mathrm{tot}} = \vartheta_{\mathrm{N}}(\rho_{\mathrm{phys}}) + \vartheta_{\mathrm{N}}(\rho_{\mathrm{app}})$ $= \arcsin \frac{1}{1+2\rho_{\mathrm{phys}}} + \arcsin \frac{1}{1+2\rho_{\mathrm{app}}} \neq \mathbf{2}\vartheta_{\mathrm{N}}(\rho_{\mathrm{app}})$	

[1] https://www.wolframalpha.com/input?i=solve+for+x%3A+x%3DR+tan%E2%81%A1%28%CE%B1+-+arcsin%E2%81%A1%281%2F%281%2B2x%29%29%29

Deflection of light:



Observed *Einstein ring = a* ***fact***

Deflection of the mind:

Senseless fabricated precious crap

Henk Reints

Henk-Reints.nl

Appendix 1: inflection points of curvature:

$$k^*(\xi) = \frac{8R^6}{(4R^4+(4R^2+1)\xi^2)^{3/2}} = 8R^6 \cdot (4R^4 + (4R^2+1)\xi^2)^{-3/2}$$

$$\frac{dk^*}{d\xi} = 8R^6 \cdot \frac{-3}{2}(4R^4 + (4R^2+1)\xi^2)^{-5/2} \cdot 2(4R^2+1)\xi$$

$$= -24R^6(4R^2+1) \cdot \xi(4R^4 + (4R^2+1)\xi^2)^{-5/2}$$

$$\frac{d^2k^*}{d\xi^2} = -24R^6(4R^2+1) \cdot \left(\xi \frac{-5}{2}(4R^4 + (4R^2+1)\xi^2)^{-7/2} \cdot 2(4R^2+1)\xi + 1(4R^4 + (4R^2+1)\xi^2)^{-5/2}\right)$$

$$= 24R^6(4R^2+1)\left(5(4R^2+1)\xi^2(4R^4 + (4R^2+1)\xi^2)^{-7/2} - (4R^4 + (4R^2+1)\xi^2)^{-5/2}\right)$$

$$= 24R^6(4R^2+1)\left(\frac{5(4R^2+1)\xi^2}{(4R^4+(4R^2+1)\xi^2)^{7/2}} - \frac{1}{(4R^4+(4R^2+1)\xi^2)^{5/2}}\right)$$

$$\boldsymbol{\frac{d^2k^*}{d\xi^2} = 0} \quad \textbf{if:} \quad \frac{5(4R^2+1)\xi^2}{(4R^4+(4R^2+1)\xi^2)^{7/2}} = \frac{1}{(4R^4+(4R^2+1)\xi^2)^{5/2}}$$

$$\frac{5(4R^2+1)\xi^2(4R^4+(4R^2+1)\xi^2)^{5/2}}{(4R^4+(4R^2+1)\xi^2)^{7/2}} = \frac{(4R^4+(4R^2+1)\xi^2)^{5/2}}{(4R^4+(4R^2+1)\xi^2)^{5/2}}$$

$$\frac{5(4R^2+1)\xi^2}{4R^4+(4R^2+1)\xi^2} = 1 \therefore 4R^4 + (4R^2+1)\xi^2 = 5(4R^2+1)\xi^2 \therefore 4R^4 = 4(4R^2+1)\xi^2$$

$$\boldsymbol{\xi_{\text{flex}} = \pm\frac{R^2}{\sqrt{4R^2+1}}}$$

Appendix 2: Standard deviation:

WANTED: $$\xi_\sigma = \sqrt{\frac{\int_0^\infty \xi^2 k^*(\xi)d\xi}{\int_0^\infty k^*(\xi)d\xi}}$$

We have: $$k^*(\xi) = \frac{8R^6}{(4R^4+(4R^2+1)\xi^2)^{3/2}}$$

We find: $\int \xi^2 k^*(\xi)d\xi =$ (WolframAlpha & HR, $\int_0^\infty$ does not converge)

$$= \frac{8R^6\left(\sqrt{4R^2+1}\sqrt{4R^4+(4R^2+1)\xi^2}\,\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - \xi(4R^2+1)\right)}{(4R^2+1)^2\sqrt{4R^4+(4R^2+1)\xi^2}}$$

and: $$\int k^*(\xi)d\xi = \frac{2R^2\xi}{\sqrt{4R^4+(4R^2+1)\xi^2}}$$

Both indefinite integrals are zero for $\xi = 0$, so we can try $\lim_{\xi\to\infty}$ of

their ratio: $$\mathfrak{R} = \frac{8R^6\left(\sqrt{4R^2+1}\sqrt{4R^4+(4R^2+1)\xi^2}\,\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - \xi(4R^2+1)\right)}{2R^2(4R^2+1)^2\xi} =$$

$$\mathfrak{R} = \frac{4R^4\left(\sqrt{\frac{4R^4}{\xi^2}+4R^2+1}\,\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - \sqrt{4R^2+1}\right)}{(4R^2+1)^{3/2}}$$

$$\Re \approx \frac{4R^4\left(\sqrt{\frac{4R^4}{\xi^2}+4R^2+1}\,\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - \sqrt{4R^2+1}\right)}{(4R^2+1)^{3/2}}$$

$$\sqrt{\frac{4R^4}{\xi^2}+4R^2+1} \approx \sqrt{4R^2+1} + \frac{2R^4}{\sqrt{4R^2+1}\,\xi^2} + \cdots$$

$$\Re \approx \frac{4R^4\left(\sqrt{4R^2+1}\,\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} + \frac{2R^4}{\sqrt{4R^2+1}\,\xi^2}\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - \sqrt{4R^2+1}\right)}{(4R^2+1)^{3/2}}$$

$$\Re \approx \frac{4R^4\left(\sqrt{4R^2+1}\left(\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - 1\right) + \frac{2R^4}{\sqrt{4R^2+1}}\cdot\frac{\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2}}{\xi^2}\right)}{(4R^2+1)^{3/2}}$$

$$\Re \approx \frac{4R^4\sqrt{4R^2+1}\left(\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - 1\right)}{(4R^2+1)^{3/2}} = \frac{4R^4\left(\operatorname{arsinh}\frac{\xi\sqrt{4R^2+1}}{2R^2} - 1\right)}{4R^2+1}$$

which does not converge.

Appendix 3: Mean curvature:

(curvature as function of light direction, avg'd over total deflection)

$$\frac{d\eta}{d\xi} = \frac{\xi}{2R\sqrt{\xi^2+R^2}} = \tan\alpha =: \delta$$

inverse: $$\xi = \frac{2R^2\delta}{\sqrt{1-4R^2\delta^2}} \quad \therefore \xi^2 = \frac{4R^4\delta^2}{1-4R^2\delta^2}$$

$$k^* = \frac{8R^6}{(4R^4+(4R^2+1)\xi^2)^{3/2}} = \frac{8R^6}{\left(4R^4+(4R^2+1)\frac{4R^4\delta^2}{1-4R^2\delta^2}\right)^{3/2}} = \frac{1}{\left(1+\frac{\delta^2(4R^2+1)}{1-4R^2\delta^2}\right)^{3/2}}$$

$$= \frac{1}{\left(\frac{1-4R^2\delta^2+4R^2\delta^2+\delta^2}{1-4R^2\delta^2}\right)^{3/2}} = \frac{1}{\left(\frac{1+\delta^2}{1-4R^2\delta^2}\right)^{3/2}} = \frac{(1-4R^2\delta^2)^{3/2}}{(1+\delta^2)^{3/2}} =: k^*(\delta)$$

we have: $$\tan\alpha = \delta \therefore \alpha = \arctan\delta \therefore \frac{d\alpha}{d\delta} = \frac{1}{1+\delta^2} \therefore d\alpha = \frac{d\delta}{1+\delta^2}$$

$$k_\mu^* = \frac{\int_0^\vartheta k^*(\alpha)d\alpha}{\vartheta} = \frac{\int_{\tan 0}^{\tan\vartheta} k^*(\delta)\frac{d\delta}{1+\delta^2}}{\vartheta} = \frac{\int_0^{\tan\vartheta}\sqrt{\frac{(1-4R^2\delta^2)^3}{(1+\delta^2)^5}}\,d\delta}{\vartheta}$$

For the indefinite integral, WolframAlpha yields a terrible behemoth including elliptic integrals and complex numbers, as well as a Taylor series expansion. We'll use the latter.

$$k_\mu^* = \frac{\left. \delta - \left(2R^2 + \frac{5}{6}\right)\delta^3 + \left(\frac{6R^4}{5} + 3R^2 + \frac{7}{8}\right)\delta^5 + \cdots \right]_0^{\tan\vartheta}}{\vartheta}$$

$$\approx \frac{\tan\vartheta - \left(2R^2 + \frac{5}{6}\right)\tan^3\vartheta + \left(\frac{6R^4}{5} + 3R^2 + \frac{7}{8}\right)\tan^5\vartheta}{\vartheta}$$

Sun: $\vartheta \approx 2 \times 10^{-6}$ rad $\quad \therefore k_\mu^* \approx \frac{\tan\vartheta}{\vartheta} \approx 1$

hence: $\xi_\mu = k^{*\mathrm{inv}}\left(k_\mu^*\right) \approx 0$

We'll round this to:

Diddly squat, nada, zilch

(but it was a nice exercise).

* * * * * * *

