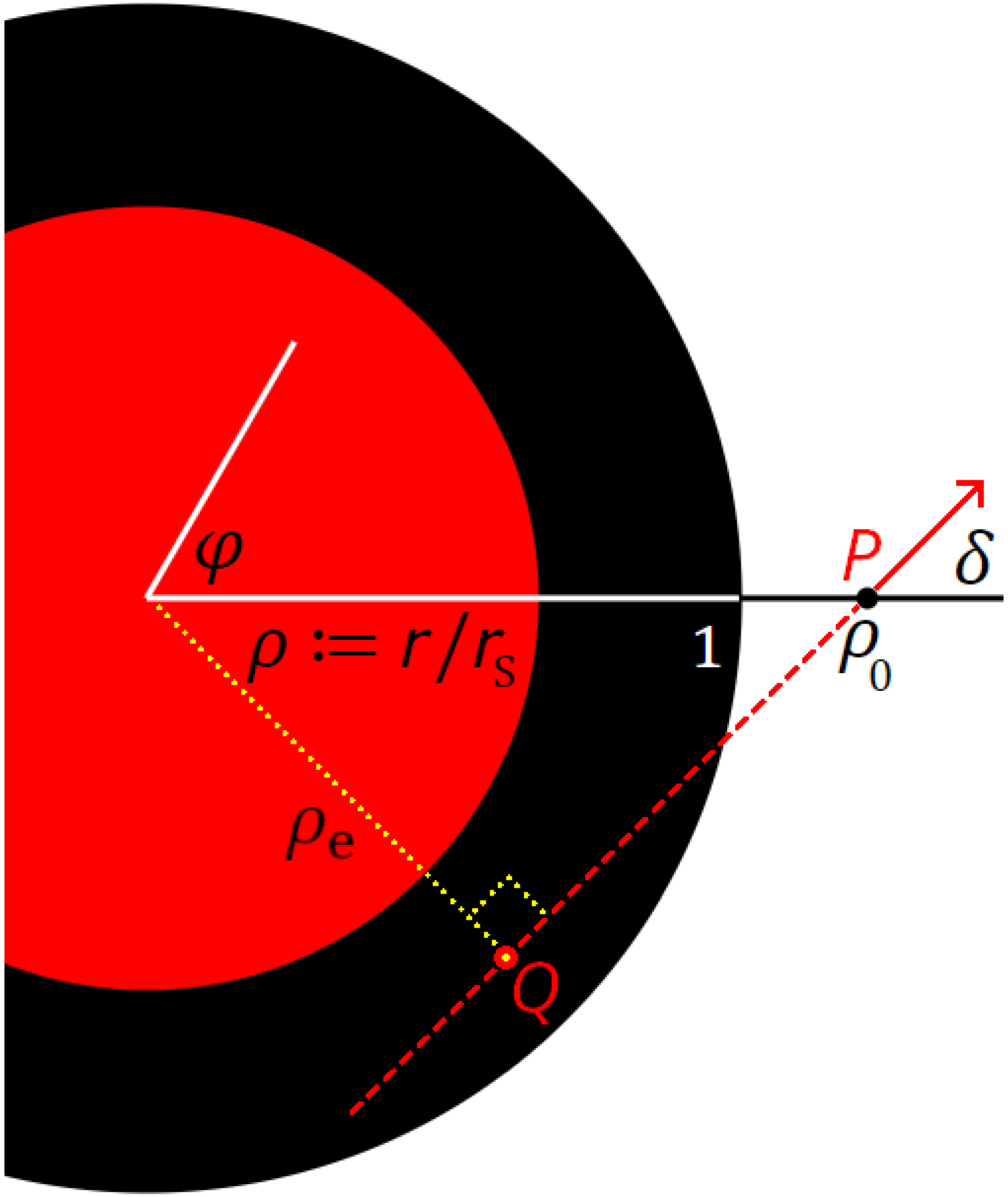

At $P: (\rho = \rho_0, \varphi = 0)$
a photon or mass
is emitted/ejected
in direction δ in
the equatorial plane
of a Schwarzschild
black hole.

What's usually called
the *impact parameter*
will now be the
ejection parameter ρ_e .

http://henk-reints.nl/astro/HR-fall-into-black-hole-slides.pdf:

Must use: $\gamma_{\text{ff}} = \dfrac{2\rho+1}{2\rho} \therefore \gamma_{\text{ff}}^2 = \dfrac{(2\rho+1)^2}{4\rho^2} = \left(1 + \dfrac{1}{2\rho}\right)^2 = 1 + \dfrac{1}{\rho} + \dfrac{1}{4\rho^2}$

instead of: $\xi = \dfrac{1}{\sqrt{1-\frac{1}{\rho}}} \quad \therefore \xi^2 = \dfrac{1}{1-\frac{1}{\rho}} = \dfrac{\rho}{\rho-1} \quad = 1 + \dfrac{1}{\rho} + \dfrac{1}{\rho^2} + \mathcal{O}\left(\dfrac{1}{\rho^3}\right)$

https://en.wikipedia.org/wiki/Schwarzschild_geodesics#Radial_motion:

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{E^2 r^4}{L^2 c^2} - \left(1 - \frac{r_S}{r}\right)\left(\frac{m^2 c^2 r^4}{L^2} + r^2\right) \qquad \text{(even if } m = 0\text{)}$$

then becomes: $r_S^2 \left(\dfrac{d\rho}{d\varphi}\right)^2 = \dfrac{E^2 r_S^4 \rho^4}{L^2 c^2} - \dfrac{4\rho^2}{(2\rho+1)^2}\left(\dfrac{m^2 c^2 r_S^4 \rho^4}{L^2} + r_S^2 \rho^2\right)$

hence: $\left(\dfrac{d\rho}{d\varphi}\right)^2 = \dfrac{E^2 r_S^2 \rho^4}{L^2 c^2} - \dfrac{4\rho^2}{(2\rho+1)^2}\left(\dfrac{m^2 c^2 r_S^2 \rho^4}{L^2} + \rho^2\right)$

or: $\left(\dfrac{d\rho}{d\varphi}\right)^2 = \dfrac{E^2 r_S^2 \rho^4}{L^2 c^2} - \dfrac{4\rho^2}{(2\rho+1)^2}\dfrac{m^2 c^2 r_S^2 \rho^4}{L^2} - \dfrac{4\rho^2}{(2\rho+1)^2}\rho^2$

so: $\left(\dfrac{d\rho}{d\varphi}\right)^2 = \rho^4 \left(\dfrac{E^2 r_S^2}{L^2 c^2} - \dfrac{4}{(2\rho+1)^2} - \dfrac{4 m^2 c^2 r_S^2 \rho^2}{L^2 (2\rho+1)^2}\right)$

We'll restrict ourselves to escaping **light**, i.e. $m = 0$,

which leaves: $$\left(\frac{d\rho}{d\varphi}\right)^2 = \rho^4\left(\frac{E^2 r_S^2}{L^2 c^2} - \frac{4}{(2\rho+1)^2}\right)$$

The aforementioned WikipediA page uses E & L as constants of motion, but isn't light subject to gravitational redshift? Doesn't that modify its energy? I'll **PRESUME** the equation holds if we use the values of E, L & $p = E/c$ in point Q in the image shown at page 1, i.e. where the ejection parameter applies.

We have: $$r_e = \rho_e r_S$$

and: $$L = p \cdot r_e = \frac{E}{c}\rho_e r_S \therefore L^2 = \frac{E^2 r_S^2}{c^2}\rho_e^2$$

yielding: $$\frac{E^2 r_S^2}{L^2 c^2} = \frac{1}{\rho_e^2}$$

hence: $$\left(\frac{d\rho}{d\varphi}\right)^2 = \rho^4\left(\frac{1}{\rho_e^2} - \frac{4}{(2\rho+1)^2}\right) = \frac{\rho^4[(2\rho+1)^2 - 4\rho_e^2]}{\rho_e^2(2\rho+1)^2}$$

therefore: $$\frac{d\rho}{d\varphi} = \pm\frac{\rho^2\sqrt{(2\rho+1)^2 - 4\rho_e^2}}{\rho_e(2\rho+1)}$$

Euclidean geometry yields: $\rho_{\mathrm{e}}=\rho_0\sin\delta$, but that is unrealistic.

We define the arc length: $d\ell=\rho d\varphi$

and then: $\dfrac{d\rho}{d\varphi}=\rho\dfrac{d\rho}{d\ell}$

Starting condition: $\left(\dfrac{d\rho}{d\ell}\right)_{\rho=\rho_0,\varphi=0}=\tan\left(\dfrac{\pi}{2}-\delta\right)=\cot\delta$

hence: $\left(\dfrac{d\rho}{d\varphi}\right)_{\rho=\rho_0,\varphi=0}=\rho_0\cot\delta$

We already found: $\dfrac{d\rho}{d\varphi}=\pm\dfrac{\rho^2\sqrt{(2\rho+1)^2-4\rho_{\mathrm{e}}^2}}{\rho_{\mathrm{e}}(2\rho+1)}$

so (use "+" since initially escaping): $\dfrac{\rho_0^2\sqrt{(2\rho_0+1)^2-4\rho_{\mathrm{e}}^2}}{\rho_{\mathrm{e}}(2\rho_0+1)}=\rho_0\cot\delta$

or: $\dfrac{\sqrt{(2\rho_0+1)^2-4\rho_{\mathrm{e}}^2}}{\rho_{\mathrm{e}}}=\dfrac{2\rho_0+1}{\rho_0}\cot\delta$

hence: $\dfrac{(2\rho_0+1)^2-4\rho_{\mathrm{e}}^2}{\rho_{\mathrm{e}}^2}=\dfrac{(2\rho_0+1)^2}{\rho_0^2}\cot^2\delta$

therefore: $$(2\rho_0+1)^2 = 4\rho_e^2 + \rho_e^2 \frac{(2\rho_0+1)^2}{\rho_0^2}\cot^2\delta$$

yielding: $$\rho_e^2 = \frac{(2\rho_0+1)^2}{4+\frac{(2\rho_0+1)^2}{\rho_0^2}\cot^2\delta} = \frac{\rho_0^2(2\rho_0+1)^2}{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}$$

which renders: $$\rho_e = \frac{\rho_0(2\rho_0+1)}{\sqrt{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}$$

Already found: $$\frac{d\rho}{d\varphi} = \pm\frac{\rho^2\sqrt{(2\rho+1)^2-4\rho_e^2}}{\rho_e(2\rho+1)}$$

Boundary condition: $$\left(\frac{d\rho}{d\varphi}\right)_{\rho=\rho_0,\varphi=0} = \rho_0\cot\delta$$

These three equations would fully describe the path of a photon that is attempting to escape from a black hole.

CHECK:

$$\frac{d\boldsymbol{\rho}}{d\boldsymbol{\varphi}}=\frac{\boldsymbol{\rho}^2\sqrt{(2\boldsymbol{\rho}+1)^2-4\boldsymbol{\rho}_e^2}}{\boldsymbol{\rho}_e(2\boldsymbol{\rho}+1)}$$

insert ρ_0 and:

$$\rho_e=\frac{\rho_0(2\rho_0+1)}{\sqrt{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}$$

$$\frac{\rho_0^2\sqrt{(2\rho_0+1)^2-\frac{4\rho_0^2(2\rho_0+1)^2}{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}}{\frac{\rho_0(2\rho_0+1)}{\sqrt{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}(2\rho_0+1)}$$

$$\frac{\rho_0^2\sqrt{(2\rho_0+1)^2-\frac{4\rho_0^2(2\rho_0+1)^2}{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}}{\frac{\rho_0(2\rho_0+1)^2}{\sqrt{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}}$$

$$\frac{\rho_0^2\sqrt{(2\rho_0+1)^2-\frac{4\rho_0^2(2\rho_0+1)^2}{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}\sqrt{4\rho_0^2+(2\rho_0+1)^2\cot^2\delta}}{\rho_0(2\rho_0+1)^2}$$

$$\frac{\rho_0\sqrt{(2\rho_0+1)^2(4\rho_0^2+(2\rho_0+1)^2\cot^2\delta)-4\rho_0^2(2\rho_0+1)^2}}{(2\rho_0+1)^2}$$

$$\frac{\rho_0\sqrt{4\rho_0^2(2\rho_0+1)^2+(2\rho_0+1)^4\cot^2\delta-4\rho_0^2(2\rho_0+1)^2}}{(2\rho_0+1)^2}=\pm\rho_0\cot\delta$$

☑ YEAH!

To be integrated:

$$\frac{d\rho}{d\varphi} = \pm\frac{\rho^2\sqrt{(2\rho+1)^2-4\rho_e^2}}{\rho_e(2\rho+1)} = \pm\sqrt{\rho^4\frac{(2\rho+1)^2-4\rho_e^2}{\rho_e^2(2\rho+1)^2}} = \pm\left(f\left(\rho(\varphi)\right)\right)^{1/2}$$

$$\frac{d}{d\varphi}f\left(\rho(\varphi)\right) = \frac{d}{d\rho}f\left(\rho(\varphi)\right)\cdot\frac{d}{d\varphi}\rho(\varphi) = \frac{d}{d\rho}\left[\rho^4\frac{(2\rho+1)^2-4\rho_e^2}{\rho_e^2(2\rho+1)^2}\right]\cdot\left[\pm\left(f\left(\rho(\varphi)\right)\right)^{1/2}\right]$$

$$\frac{d^2\rho}{d\varphi^2} = \pm\frac{1}{2}\left(f\left(\rho(\varphi)\right)\right)^{-1/2}\cdot\frac{d}{d\varphi}f\left(\rho(\varphi)\right)$$

$$\frac{d^2\rho}{d\varphi^2} = \left[\pm\frac{1}{2}\left(f\left(\rho(\varphi)\right)\right)^{-1/2}\right]\cdot\frac{d}{d\rho}\left[\rho^4\frac{(2\rho+1)^2-4\rho_e^2}{\rho_e^2(2\rho+1)^2}\right]\cdot\left[\pm\left(f\left(\rho(\varphi)\right)\right)^{1/2}\right]$$ (same sign twice)

$$\frac{d^2\rho}{d\varphi^2} = \frac{1}{2}\frac{d}{d\rho}\left[\rho^4\frac{(2\rho+1)^2-4\rho_e^2}{\rho_e^2(2\rho+1)^2}\right] = 2\rho^3\left(\frac{1}{\rho_e^2}-\frac{4(\rho+1)}{(2\rho+1)^3}\right)$$

(differentiated by hand, checked with WolframAlpha)

We can now use the 2nd-order Taylor approximation:

$\Delta\rho = \rho'(\varphi)\Delta\varphi + \frac{1}{2}\rho''(\varphi)[\Delta\varphi]^2$, then use sign of $\Delta\rho$ for next $\rho'(\varphi)$

$$\rho(\varphi+\Delta\varphi) = \rho(\varphi) + \Delta\rho + \mathcal{O}([\Delta\varphi]^3)$$

Reciprocal D.E.: $$\frac{d\varphi}{d\rho} = \frac{\rho_\mathrm{e}(2\rho+1)}{\rho^2\sqrt{(2\rho+1)^2-4\rho_\mathrm{e}^2}}$$ (if $0 < \delta < \frac{\pi}{2}$)

WolframAlpha renders:

$$\varphi(\rho) = C - \frac{\rho_\mathrm{e}\sqrt{(2\rho+1)^2-4\rho_\mathrm{e}^2}}{(1-4\rho_\mathrm{e}^2)\rho} + \frac{8\rho_\mathrm{e}^3\,\mathrm{artanh}\frac{2\rho+1-4\rho_\mathrm{e}^2}{\sqrt{1-4\rho_\mathrm{e}^2}\sqrt{(2\rho+1)^2-4\rho_\mathrm{e}^2}}}{(1-4\rho_\mathrm{e}^2)^{3/2}}$$

where C follows from boundary condition: $\varphi(\rho_0) = 0$

This should yield the same trajectory

as by solving the D.E. with $\frac{d\rho}{d\varphi}$ and $\frac{d^2\rho}{d\varphi^2}$,

but since it is an exact solution,

it should be more accurate.

However, a numerical evaluation yields

nothing useful for this exact solution.

But numerically solving the D.E. *does* render a useful result:

in accordance with

http://henk-reints.nl/astro/HR-fall-into-black-hole-slides.pdf,

all light in all directions would be able to escape, even if $\rho_0 < 1$ is very near zero!

But according to observations, the thing is black!

As stated in the above BH document, it must be that the radiation pressure cannot surpass the pressure asymptote explained in

http://henk-reints.nl/astro/HR-Schwarzschild-interior.pdf