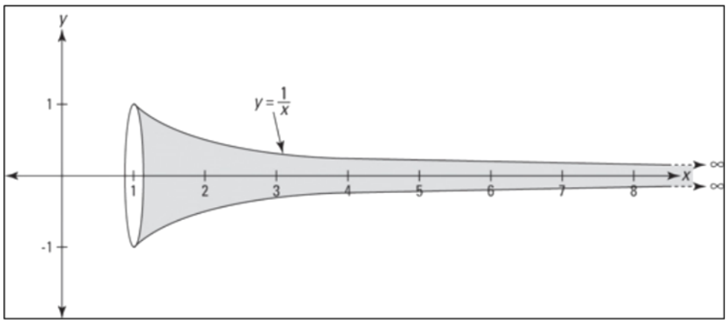
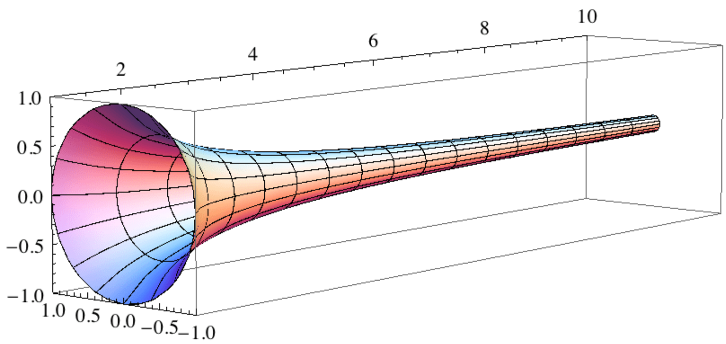
The archangel Gabriel plays a horn:

which mathemagicians like to represent by:

$$f(x) = \frac{1}{x}$$

(for x from 1 to a where a is the length of the horn minus 1), rotated around the x-axis:





It surrounds a volume of: $V(a) = \int_1^a \frac{\pi}{x^2} dx = \left.\frac{-\pi}{x}\right]_1^a = \frac{-\pi}{a} - \frac{-\pi}{1} = \pi\left(1 - \frac{1}{a}\right)$

and it has a surface area equal to: $A(a) = 2\pi \int_1^a \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx > 2\pi \int_1^a \frac{1}{x} dx = 2\pi \ln a$

(https://en.wikipedia.org/wiki/Gabriel%27s_Horn, https://en.wikipedia.org/wiki/Solid_of_revolution, https://en.wikipedia.org/wiki/Surface_of_revolution)

The longer the horn gets, it approaches: $V = \lim_{a\to\infty} V(a) = \pi$

and: $A = \lim_{a\to\infty} A(a) = \infty$

so Gabriel's mathemagical horn of infinite length has an infinite surface area, but a finite volume!

This is also called the painter's paradox. Because $V = \pi$ I would rather call it the πnter's πradox... A finite amount of πnt would fully πnt the infinite surface area, whilst the same amount of πnt would still remain, not touching the horn's surface.

But mathemagically, the thickness of the layer of πnt would be nought and then the total volume of the layer would be $0 \times \infty$, which like $\frac{0}{0}$ can have any value, so it may well be diddly squat.

Physically, the thickness of the πnt layer would be at least 1 atom, and as the horn gets longer, its diameter will become smaller, so the πnting will just stop at a given length. Moreover, the cosmos does not contain enough matter for an infinitely long horn. Physical infinity is a myth.

I presume you find it not strange at all that the surface area under a curve can be finite whilst the length of the curve is infinite, like for example:

$$\int_0^{\infty} e^{-x}\,dx = 1 \text{ and the Gaussian integral: } \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

It is just a shape with a finite surface and an infinite circumference, and in the same way the horn's volume can be finite whilst both its length and surface area are infinite.

Exact surface area:

on https://www.integral-calculator.com/ we find (after clicking the "Simplify" button):

$$\frac{A(a)}{2\pi} = \int_1^a \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = \frac{\ln\left(\frac{\sqrt{a^4+1}+a^2}{a^2}\right) - \ln\left(\frac{\sqrt{a^4+1}-a^2}{a^2}\right) - \ln(\sqrt{2}+1) + \ln(\sqrt{2}-1)}{4} - \frac{\sqrt{a^4+1}}{2a^2} + \frac{1}{\sqrt{2}}$$

Let's simplify this even further (please note: simplification is not necessarily a simple process...).

It equals: $$\frac{A(a)}{2\pi} = \frac{1}{2}\sqrt{2} - \frac{1}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \frac{1}{2}\sqrt{1+\frac{1}{a^4}} + \frac{1}{4}\ln\left(\frac{\sqrt{a^4+1}+a^2}{\sqrt{a^4+1}-a^2}\right)$$

Now: $$\frac{\sqrt{a^4+1}+a^2}{\sqrt{a^4+1}-a^2} = \frac{\sqrt{a^4+1}+a^2}{\sqrt{a^4+1}-a^2}\cdot\frac{\sqrt{a^4+1}+a^2}{\sqrt{a^4+1}+a^2} = \frac{\left(\sqrt{a^4+1}+a^2\right)^2}{\sqrt{a^4+1}^2-(a^2)^2} = \frac{\left(\sqrt{a^4+1}+a^2\right)^2}{a^4+1-a^4} = \left(\sqrt{a^4+1}+a^2\right)^2$$

and: $$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}^2+2\sqrt{2}+1}{\sqrt{2}^2-1^2} = 3+2\sqrt{2}$$

hence: $$\frac{A(a)}{2\pi} = \frac{1}{2}\sqrt{2} - \frac{1}{4}\ln\left(3+2\sqrt{2}\right) - \frac{1}{2}\sqrt{1+\frac{1}{a^4}} + \frac{1}{4}\ln\left(\sqrt{a^4+1}+a^2\right)^2$$

which equals: $$\frac{A(a)}{2\pi} = \ln e^{\frac{1}{2}\sqrt{2}} - \ln\sqrt[4]{3+2\sqrt{2}} - \ln e^{\frac{1}{2}\sqrt{1+\frac{1}{a^4}}} + \ln\sqrt{a^2+\sqrt{a^4+1}}$$

therefore: $$A(a) = 2\pi\ln\frac{e^{\frac{1}{2}\left(\sqrt{2}-\sqrt{1+\frac{1}{a^4}}\right)}\cdot\sqrt{a^2+\sqrt{a^4+1}}}{\sqrt[4]{3+2\sqrt{2}}}$$

For large values of a this can be approximated

as: $$A(a\to\infty) \approx 2\pi\ln\frac{e^{\frac{1}{2}(\sqrt{2}-\sqrt{1})}\cdot\sqrt{2a^2}}{\sqrt[4]{3+2\sqrt{2}}} = 2\pi\ln\frac{e^{\frac{1}{2}(\sqrt{2}-1)}\cdot\sqrt{2}\cdot a}{\sqrt[4]{3+2\sqrt{2}}} \approx 2\pi\ln(1.1196\cdot a)$$

which is indeed greater than the aforementioned $2\pi\ln a$.