The derivation below applies only to two *velocities* along a straight line!

Dimensionless *velocity*: $\beta = \frac{v}{c}$

Lorentz factor: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $(\gamma \geq 1)$

so: $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$

Define: $\Gamma = \frac{1}{\gamma}$

then: $\Gamma = \sqrt{1 - \beta^2}$ $(0 < \Gamma \leq 1)$

and: $\beta = \pm\sqrt{1 - \Gamma^2}$

so: $\beta^2 + \Gamma^2 = 1$ (which in fact is the Pythagorean theorem)

<u>**Superposition of *velocities***</u> (assuming $\beta_1 \geq \beta_2 \geq 0$):

$\beta_0 = \frac{\beta_1 \pm \beta_2}{1 \pm \beta_1 \cdot \beta_2}$ using the same sign twice

so: $$1 - \Gamma_0^2 = \left(\frac{\sqrt{1-\Gamma_1^2} \pm \sqrt{1-\Gamma_2^2}}{1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}}\right)^2$$

yielding: $$\Gamma_0^2 = 1 - \left(\frac{\sqrt{1-\Gamma_1^2} \pm \sqrt{1-\Gamma_2^2}}{1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}}\right)^2 = 1 - \frac{\left(\sqrt{1-\Gamma_1^2} \pm \sqrt{1-\Gamma_2^2}\right)^2}{\left(1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}\right)^2}$$

$$= \frac{\left(1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}\right)^2 - \left(\sqrt{1-\Gamma_1^2} \pm \sqrt{1-\Gamma_2^2}\right)^2}{\left(1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}\right)^2}$$

therefore: $$\Gamma_0 = \frac{\sqrt{\left(1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}\right)^2 - \left(\sqrt{1-\Gamma_1^2} \pm \sqrt{1-\Gamma_2^2}\right)^2}}{1 \pm \sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}}$$

Substitute: $A = 1 \pm \sqrt{(1 - \Gamma_1^2)(1 - \Gamma_2^2)}$

and: $B = \sqrt{1 - \Gamma_1^2} \pm \sqrt{1 - \Gamma_2^2}$

yielding: $\Gamma_0 = \frac{\sqrt{A^2 - B^2}}{A}$

Now we've got: $A^2 = \left(1 \pm \sqrt{(1 - \Gamma_1^2)(1 - \Gamma_2^2)}\right)^2$

and by using: $(a \pm b)^2 = a^2 \pm 2ab + b^2$

we get: $A^2 = 1 \pm 2\sqrt{(1 - \Gamma_1^2)(1 - \Gamma_2^2)} + (1 - \Gamma_1^2)(1 - \Gamma_2^2)$

By writing out the last term thereof: $(1 - \Gamma_1^2)(1 - \Gamma_2^2) = 1 - \Gamma_1^2 - \Gamma_2^2 + \Gamma_1^2\Gamma_2^2$
and adding this 1 to the one we've already got,

we obtain: $A^2 = 2 \pm 2\sqrt{(1 - \Gamma_1^2)(1 - \Gamma_2^2)} - \Gamma_1^2 - \Gamma_2^2 + \Gamma_1^2\Gamma_2^2$

We also have: $B^2 = \left(\sqrt{1 - \Gamma_1^2} \pm \sqrt{1 - \Gamma_2^2}\right)^2$

once again using: $(a \pm b)^2 = a^2 \pm 2ab + b^2$

this gives: $B^2 = 1 - \Gamma_1^2 \pm 2\sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)} + 1 - \Gamma_2^2$

so: $B^2 = 2 \pm 2\sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)} - \Gamma_1^2 - \Gamma_2^2$

We already got: $A^2 = 2 \pm 2\sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)} - \Gamma_1^2 - \Gamma_2^2 + \Gamma_1^2\Gamma_2^2$

So: $A^2 - B^2 = \Gamma_1^2\Gamma_2^2$

hence: $\Gamma_0 = \frac{\sqrt{A^2-B^2}}{A} = \frac{\sqrt{\Gamma_1^2\Gamma_2^2}}{1\pm\sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}} = \frac{\Gamma_1\Gamma_2}{1\pm\sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}}$

yielding: $\gamma_0 = \frac{1}{\Gamma_0} = \frac{1\pm\sqrt{(1-\Gamma_1^2)(1-\Gamma_2^2)}}{\Gamma_1\Gamma_2} = \frac{1\pm\sqrt{\left(1-\frac{1}{\gamma_1^2}\right)\left(1-\frac{1}{\gamma_2^2}\right)}}{\frac{1}{\gamma_1\gamma_2}}$

$$= \gamma_1\gamma_2 \pm \gamma_1\gamma_2\sqrt{\left(1-\frac{1}{\gamma_1^2}\right)\left(1-\frac{1}{\gamma_2^2}\right)} = \gamma_1\gamma_2 \pm \sqrt{\gamma_1^2\left(1-\frac{1}{\gamma_1^2}\right)\gamma_2^2\left(1-\frac{1}{\gamma_2^2}\right)}$$

Therefore: $\boldsymbol{\gamma_0 = \gamma_1\gamma_2 \pm \sqrt{(\gamma_1^2-1)(\gamma_2^2-1)}}$

For $\boldsymbol{\gamma_1 = \gamma_2}$ we get: $\gamma_0 = \gamma_1^2 + \sqrt{(\gamma_1^2-1)(\gamma_1^2-1)} = \gamma_1^2 + (\gamma_1^2 - 1)$

so: $\boldsymbol{\gamma_0 = 2\gamma_1^2 - 1}$

An approximation of the additive superposition of unequal but not too different values is:

$$\boldsymbol{\gamma_0 \approx 2\gamma_1\gamma_2 - 1}$$