Newton's 2nd law says that acceleration $a = F/m$ equals the specific force (i.e. force per mass), exerted on a body. Of course, the same applies to *gravitational* acceleration g .

Gravitation by a ring

An infinitesimal ring with radius r and uniform linear density has, at position p (distance from center), (see: http://henk-reints.nl/astro/HR-Dark-matter-slideshow.pdf , p.20 [as of 2024-04-18]):

$$dg_{\text{ring}} = G\rho r \int_0^{2\pi} \frac{p - r\cos\varphi}{(p^2+r^2-2pr\cos\varphi)^{3/2}} d\varphi \; dr$$

where $\rho =$ uniform *surface* density of full disk that ultimately results from integration over all rings:

$$\rho = \frac{M}{\pi R_{\text{disk}}^2}$$

We define: $$g_{\text{ring}}^* \coloneqq \frac{g_{\text{ring}}}{G\rho} = \frac{\pi R_{\text{disk}}^2 g_{\text{ring}}}{GM} \therefore g_{\text{ring}} = \frac{GM}{\pi R_{\text{disk}}^2} g_{\text{ring}}^*$$

hence: $$\frac{dg_{\text{ring}}^*}{dr} = r \int_0^{2\pi} \frac{p - r\cos\varphi}{(p^2+r^2-2pr\cos\varphi)^{3/2}} d\varphi$$

We simplify it by first complicating it:

$$\xi \coloneqq \frac{p - r\cos\varphi}{(p^2+r^2-2pr\cos\varphi)^{3/2}} = \frac{r\left(\frac{p}{r} - \frac{r}{r}\cos\varphi\right)}{(r^2)^{3/2}\left(\frac{p^2}{r^2}+\frac{r^2}{r^2}-\frac{2pr\cos\varphi}{r^2}\right)^{3/2}}$$

We define: $$q \coloneqq p/r$$

yielding: $$\xi = \frac{r(q-\cos\varphi)}{r^3(q^2+1-2q\cos\varphi)^{3/2}} = \frac{1}{r^2} \cdot \frac{q-\cos\varphi}{(q^2+1-2q\cos\varphi)^{3/2}}$$

which renders: $$\frac{dg_{\text{ring}}^*}{dr} = r\int_0^{2\pi} \xi \, d\varphi = r \frac{1}{r^2} \int_0^{2\pi} \frac{q-\cos\varphi}{(q^2+1-2q\cos\varphi)^{3/2}} d\varphi$$

or: $$\frac{dg_{\text{ring}}^*}{dr} = \frac{1}{r}\int_0^{2\pi} \frac{q-\cos\varphi}{(q^2+1-2q\cos\varphi)^{3/2}} d\varphi = \frac{1}{r}\psi(q) = \frac{1}{r}\psi\left(\frac{p}{r}\right)$$

i.e.: $$\boldsymbol{\frac{dg_{\text{ring}}}{dr} = \frac{GM}{\pi r_{\text{disk}}^2} \cdot \frac{1}{r} \psi\left(\frac{p}{r}\right)}$$

where: $$\boldsymbol{\psi(q) \coloneqq \int_0^{2\pi} \frac{q-\cos\varphi}{(q^2+1-2q\cos\varphi)^{3/2}} d\varphi}$$

At great distance, ψ becomes "normal" gravity $\propto 1/p^2$ (note: p is distance to center, r is ring radius), we have: $$\lim_{q\to\infty}(q - \cos\varphi) = q$$

as well as: $$\lim_{q\to\infty}(q^2 + 1 - 2q\cos\varphi) = q^2$$

therefore: $$\boldsymbol{\lim_{q\to\infty} \psi(q)} = \frac{2q}{(q^2)^{3/2}}\int_0^{\pi} d\varphi = \boldsymbol{\frac{2\pi}{q^2} = \frac{2\pi r^2}{p^2}}$$

Uniform disk

In orde to create a disk, we must integrate rings:

$$g_{\text{disk}}^*(p; R_{\text{disk}}) = \int_0^{R_{\text{disk}}} dg_{\text{ring}}^* = \int_0^{R_{\text{disk}}} \frac{1}{r}\psi\left(\frac{p}{r}\right) dr$$

which should be the dimensionless gravitational acceleration = specific force at position $p > R_{\text{disk}}$ from the center of a uniform massive disk.

With: $q = \frac{p}{r} \therefore r = \frac{p}{q} \therefore \frac{dr}{dq} = \frac{-p}{q^2} \therefore dr = \frac{-p}{q^2} dq$ and: $\frac{1}{r} = \frac{q}{p}$

this becomes: $$\boldsymbol{g_{\text{disk}}^*(p; R_{\text{disk}})} = \int_{\infty}^{p/R_{\text{disk}}} \frac{q}{p}\psi(q)\frac{-p}{q^2} dq = \boldsymbol{\int_{p/R_{\text{disk}}}^{\infty} \frac{\psi(q)}{q} dq}$$

or, with: $$q_{\text{disk}} \coloneqq \frac{p}{R_{\text{disk}}}$$

we obtain: $$\boldsymbol{g^*_{\text{disk}}(q_{\text{disk}}) = \int_{q_{\text{disk}}}^{\infty} \frac{\psi(q)}{q}\, dq}$$

We also have: $$\boldsymbol{g_{\text{disk}}(p; R_{\text{disk}}) = \frac{GM}{\pi R^2_{\text{disk}}} g^*_{\text{disk}}(p; R_{\text{disk}})}$$

as well as: $$\boldsymbol{g_{\text{disk}}(q_{\text{disk}}) = \frac{GM}{\pi R^2_{\text{disk}}} g^*_{\text{disk}}(q_{\text{disk}})}$$

Solving $\boldsymbol{\psi(q) = \int_0^{2\pi} \frac{q - \cos\varphi}{(q^2+1-2q\cos\varphi)^{3/2}}\, d\varphi}$

WolframAlpha yields the indefinite integral:

$$\Psi(q,x) = \int \frac{q-\cos x}{(q^2+1-2q\cos x)^{3/2}}\, dx$$

as:

$$= \frac{(q^2-1)\sqrt{\frac{q^2+1-2q\cos x}{(q-1)^2}}F\left(\frac{x}{2}\middle|\frac{-4q}{(q-1)^2}\right) + (q-1)^2\sqrt{\frac{q^2+1-2q\cos x}{(q-1)^2}}E\left(\frac{x}{2}\middle|\frac{-4q}{(q-1)^2}\right) + 2q\sin x}{q(q^2-1)\sqrt{q^2+1-2q\cos x}}$$

where: $E(x|m)$ is the elliptic integral of the 2nd kind with parameter $m = k^2$

and: $F(x|m)$ is the elliptic integral of the 1st kind with parameter $m = k^2$

we rewrite: $$\Psi(q,x) = \frac{(q+1)(q-1)\sqrt{\frac{q^2+1-2q\cos x}{(q-1)^2}}F\left(\frac{x}{2}\middle|\frac{-4q}{(q-1)^2}\right) + (q-1)^2\sqrt{\frac{q^2+1-2q\cos x}{(q-1)^2}}E\left(\frac{x}{2}\middle|\frac{-4q}{(q-1)^2}\right) + 2q\sin x}{q(q+1)(q-1)\sqrt{q^2+1-2q\cos x}}$$

For $q > 1$: $$\Psi(q,x) = \frac{(q+1)\sqrt{q^2+1-2q\cos x}\,F\left(\frac{x}{2}\middle|\frac{-4q}{(q-1)^2}\right) + (q-1)\sqrt{q^2+1-2q\cos x}\,E\left(\frac{x}{2}\middle|\frac{-4q}{(q-1)^2}\right) + 2q\sin x}{q(q+1)(q-1)\sqrt{q^2+1-2q\cos x}}$$

We need: $\Psi(q,2\pi) - \Psi(q,0)$

we have: $$\Psi(q,2\pi) = \frac{(q+1)\sqrt{q^2+1-2q}\,F\left(\pi\middle|\frac{-4q}{(q-1)^2}\right) + (q-1)\sqrt{q^2+1-2q}\,E\left(\pi\middle|\frac{-4q}{(q-1)^2}\right)}{q(q+1)(q-1)\sqrt{q^2+1-2q}}$$

$$= \frac{(q+1)F\left(\pi\middle|\frac{-4q}{(q-1)^2}\right) + (q-1)E\left(\pi\middle|\frac{-4q}{(q-1)^2}\right)}{q(q+1)(q-1)} = \frac{F\left(\pi\middle|\frac{-4q}{(q-1)^2}\right)}{q(q-1)} + \frac{E\left(\pi\middle|\frac{-4q}{(q-1)^2}\right)}{q(q+1)}$$

and: $$\Psi(q,0) = \frac{F\left(0\middle|\frac{-4q}{(q-1)^2}\right)}{q(q-1)} + \frac{E\left(0\middle|\frac{-4q}{(q-1)^2}\right)}{q(q+1)} = 0$$ (WolframAlpha)

yielding: $$\boldsymbol{\psi_{\text{approx}}(q) = \frac{F\left(\pi\middle|\frac{-4q}{(q-1)^2}\right)}{q(q-1)} + \frac{E\left(\pi\middle|\frac{-4q}{(q-1)^2}\right)}{q(q+1)}}$$

For which WolframAlpha gives a Laurent series at $q = \infty$.

Repeatedly clicking on [More terms] until no more terms are added

yields: $$\psi_{\text{approx}}(q) = \sum_{k=0}^{39} \frac{c_k \pi}{q^{2k}} + \mathcal{O}\left(\frac{1}{q^{80}}\right)$$ (WolframAlpha says $\mathcal{O}\left(\frac{1}{q^{79}}\right)$)

hence: $$\frac{\psi_{\text{approx}}(q)}{q} = \sum_{k=0}^{k_{\max}} \frac{c_k \pi}{q^{2k+1}}$$

where: c_k = see **Appendix I** @page 5.

We now have something that we can easily integrate, although it has quite many terms.

With $k_{\max} = 39$, it is not really very accurate for q very close to 1. A numerical analysis yields: for $q = 1.01$, we have $\Delta\psi \approx -45\%$, which rapidly drops to $|\Delta\psi| < 1\%$ for $q \geq 1.06$ and for $q \geq 2.36$, it has $|\langle\Delta\psi\rangle| < 10^{-15}$ (smoothed average: $\langle\Delta\psi\rangle_n = 0.75\langle\Delta\psi\rangle_{n-1} + 0.25\Delta\psi_n$).
Since $\psi(q)$ has a vertical asymptote at $q = 1$, I also investigated $\psi_{\text{approx}}(s)$ where $s = q - 1$.
WolframAlpha *did* render a series, but it appeared ***WAY*** less accurate for s near zero.

Uniform disk — continued

We found: $g^*_{\text{disk,approx}}(p;R_{\text{disk}}) = \int_{p/R_{\text{disk}}}^{\infty} \frac{\psi_{\text{approx}}(q)}{q} dq = \int_{p/R_{\text{disk}}}^{\infty} \sum_{k=0}^{k_{\max}} \frac{c_k \pi}{q^{2k+1}} dq$

or: $$g^*_{\text{disk,approx}}(p;R_{\text{disk}}) = \pi \sum_{k=0}^{k_{\max}} \int_{p/R_{\text{disk}}}^{\infty} \frac{c_k}{q^{2k+1}} dq = \pi \sum_{k=0}^{k_{\max}} \frac{-c_k}{2kq^{2k}}\Big]_{p/R_{\text{disk}}}^{\infty}$$

i.e.: $$g^*_{\text{disk,approx}}(p;R_{\text{disk}}) = \pi \sum_{k=0}^{k_{\max}} \frac{c_k}{2kq^{2k}}\Big]_{\infty}^{p/R_{\text{disk}}} = \pi \sum_{k=0}^{k_{\max}} \frac{c_k}{2k(p/R_{\text{disk}})^{2k}}$$

hence: $$\boldsymbol{g^*_{\text{disk,approx}}(p;R_{\text{disk}}) = \pi \sum_{k=0}^{k_{\max}} \frac{c_k R_{\text{disk}}^{2k}}{2kp^{2k}}}$$

or, with: $$q_{\text{disk}} \coloneqq \frac{p}{R_{\text{disk}}}$$

we find: $$\boldsymbol{g^*_{\text{disk,approx}}(q_{\text{disk}}) = \pi \sum_{k=0}^{k_{\max}} \frac{c_k}{2kq_{\text{disk}}^{2k}}}$$

which is fully dimensionless.

With: $$C_{\text{k}} \coloneqq \frac{c_{\text{k}}\pi}{2k}$$ see **Appendix II** @page 6

we would encounter a problem for $k = 0$, but the series actually starts with $k = 1$,

so we can simply equate: $$C_0 = 0$$

and obtain: $$\boldsymbol{g^*_{\text{disk,approx}}(q_{\text{disk}}) = \sum_{k=0}^{k_{\max}} \frac{C_k}{q_{\text{disk}}^{2k}}}$$

which is the simplest form I can find.

It renders: $$\boldsymbol{g_{\text{disk,approx}}(q_{\text{disk}}) = \frac{GM}{\pi R_{\text{disk}}^2} \sum_{k=0}^{k_{\max}} \frac{C_k}{q_{\text{disk}}^{2k}}}$$

Gravitational potential energy inside a uniform disk

We create a ring of mass dm by letting it precipitate in free fall from infinity onto a disk's edge.

It feels a force: $$dF = g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) dm$$

where $g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}})$ is the specific gravitational force (i.e. acceleration) at freefall position p_{ff} by the disk built up sofar (by showers that came down earlier).

The radius of this ring is: $$r_{\text{ring}} = r_{\text{disk}}$$

For a ring, we have: $$dm = \rho \cdot 2\pi r_{\text{ring}} dr_{\text{ring}} = \frac{M}{\pi R_{\text{disk}}^2} \cdot 2\pi r_{\text{disk}} dr_{\text{disk}} = \frac{2M}{R_{\text{disk}}^2} r_{\text{disk}} dr_{\text{disk}}$$

hence: $$dF = \frac{2M}{R_{\text{disk}}^2} g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) r_{\text{disk}} dr_{\text{disk}}$$

The work done during an infinitesimal drop is: $$dW = dF\, dp_{\text{ff}} = \left(\frac{2M}{R_{\text{disk}}^2} g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) r_{\text{disk}} dr_{\text{disk}}\right) dp_{\text{ff}}$$

Integration over the entire fall yields the released potential energy of the shower: $$dE_{\text{pot,shower}} = \int_{\infty}^{r_{\text{disk}}} dW$$
$$= \int_{\infty}^{r_{\text{disk}}} \left(\frac{2M}{R_{\text{disk}}^2} g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) r_{\text{disk}} dr_{\text{disk}}\right) dp_{\text{ff}}$$

hence: $$dE_{\text{pot,shower}} = \frac{2M}{R_{\text{disk}}^2} \left(\int_{\infty}^{r_{\text{disk}}} g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) dp_{\text{ff}}\right) r_{\text{disk}} dr_{\text{disk}}$$

Now we have to add up all showers in order to obtain a massive disk.

$$E_{\text{pot,disk}} = \int_0^{R_{\text{disk}}} dE_{\text{pot,shower}}$$

hence: $$E_{\text{pot,disk}} = \frac{2M}{R_{\text{disk}}^2} \int_0^{R_{\text{disk}}} \left(\int_{\infty}^{r_{\text{disk}}} g_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) dp_{\text{ff}}\right) r_{\text{disk}} dr_{\text{disk}}$$

We had defined: $$g = \frac{GM}{\pi R_{\text{disk}}^2} g^*$$

which renders: $$E_{\text{pot,disk}} = \frac{2GM^2}{\pi R_{\text{disk}}^4} \int_0^{R_{\text{disk}}} \left(\int_{\infty}^{r_{\text{disk}}} g^*_{\text{disk}}(p_{\text{ff}}; r_{\text{disk}}) dp_{\text{ff}}\right) r_{\text{disk}} dr_{\text{disk}}$$

We're nearly there!

We define: $$\ell(r_\text{disk}) \coloneqq \int_\infty^{r_\text{disk}} g^*_\text{disk}(p_\text{ff}; r_\text{disk})dp_\text{ff}$$ [LENGTH]

yielding: $$E_\text{pot,disk} = \frac{2GM^2}{\pi R^4_\text{disk}} \int_0^{R_\text{disk}} \ell(r_\text{disk}) r_\text{disk} dr_\text{disk}$$

We had found: $$\boldsymbol{g^*_\text{disk}(q_\text{disk}) = \int_{q_\text{disk}}^\infty \frac{\psi(q)}{q} dq = \int_{q_\text{disk}}^\infty \frac{1}{q} \int_0^{2\pi} \frac{q - \cos\varphi}{(q^2 + 1 - 2q\cos\varphi)^{3/2}} d\varphi\, dq}$$

as well as: $$\boldsymbol{g^*_\text{disk,approx}(q_\text{disk}) = \sum_{k=0}^{k_\text{max}} \frac{C_k}{q^{2k}_\text{disk}}}$$

so we change the variable from p_ff to $q_\text{ff} = p_\text{ff}/r_\text{disk}$, hence $dp_\text{ff} = r_\text{disk} dq_\text{ff}$, the integration boundaries change to ∞ and 1, and $g^*_\text{disk}(p_\text{ff}; r_\text{disk})$ becomes $g^*_\text{disk}(q_\text{ff})$:

therefore: $$E_\text{pot,disk} = \frac{2GM^2}{\pi R^4_\text{disk}} \int_0^{R_\text{disk}} \left(\int_\infty^1 g^*_\text{disk}(q_\text{ff}) dq_\text{ff} \right) r^2_\text{disk} dr_\text{disk}$$

and: $$\ell(r_\text{disk}) = \int_\infty^1 g^*_\text{disk}(q_\text{ff}) r_\text{disk} dq_\text{ff} = r_\text{disk} \int_\infty^1 g^*_\text{disk}(q_\text{ff}) dq_\text{ff}$$

We define: $$\ell^*(r_\text{disk}) \coloneqq \frac{\ell(r_\text{disk})}{r_\text{disk}} = \int_\infty^1 g^*_\text{disk}(q_\text{ff}) dq_\text{ff}$$

and then: $$E_\text{pot,disk} = \frac{2GM^2}{\pi R^4_\text{disk}} \int_0^{R_\text{disk}} \ell^*(r_\text{disk}) r^2_\text{disk} dr_\text{disk}$$

Let's concentrate on $g^*_\text{disk,approx}(q_\text{disk})$:

$$\ell^*(r_\text{disk}) = \int_\infty^1 \sum_{k=0}^{k_\text{max}} \frac{C_k}{q^{2k}_\text{ff}} dq_\text{ff} = \int_\infty^1 \sum_{k=0}^{k_\text{max}} \frac{C_k}{q^{2k}_\text{ff}} dq_\text{ff}$$

$$= \sum_{k=0}^{k_\text{max}} \int_\infty^1 \frac{C_k}{q^{2k}_\text{ff}} dq_\text{ff} = \sum_{k=0}^{k_\text{max}} \left(\left[\frac{C_k}{(1-2k) q^{2k-1}_\text{ff}} \right]_\infty^1 \right) = \sum_{k=0}^{k_\text{max}} \frac{C_k}{1-2k} =: \mathcal{K}_{k_\text{max}}$$

which brings us to:

$$E_\text{pot,disk} = \mathcal{K}_{k_\text{max}} \frac{2GM^2}{\pi R^4_\text{disk}} \int_0^{R_\text{disk}} r^2_\text{disk} dr_\text{disk} = \mathcal{K}_{k_\text{max}} \frac{2GM^2}{\pi R^4_\text{disk}} \cdot \frac{R^3_\text{disk}}{3}$$

and there we have it: $$\boldsymbol{E_\text{pot,disk} = \mathcal{K}_{k_\text{max}} \frac{2GM^2}{3\pi R_\text{disk}}}$$

Numerical evaluation: $$\boldsymbol{\mathcal{K}_{39}} \approx -3.974\,44$$

I *ESTIMATE*: $$\lim_{k_\text{max}\to\infty} \mathcal{K}_{k_\text{max}} = \boldsymbol{\mathcal{K}_\infty \approx -4} \qquad \textbf{???}$$

yielding: $$\boldsymbol{\left| E_\text{pot,disk} \right| \approx \frac{8GM^2}{3\pi R_\text{disk}} \approx 0.845 \frac{GM^2}{R_\text{disk}}}$$

For a uniform sphere, we have: $$\boldsymbol{\left| E_\text{pot,sphere} \right| = \frac{3GM^2}{5R_\text{sphere}} = 0.6 \frac{GM^2}{R_\text{sphere}}}$$

Their ratio is about: $$\frac{8}{3\pi} \Big/ \frac{3}{5} = \frac{40}{9\pi} \approx \mathbf{1.415}$$

A gravitationally collapsing disk releases roughly 1½ times more energy than a collapsing sphere!

In a relative sense, the core of a planet must be hotter that that of a protostar!

See also:

http://henk-reints.nl/astro/HR-BH-temperature.pdf ,

where I substantiate that **Earth's core** might
very plausibly be **well over 100 kilokelvin**
instead of the generally accepted **6000 kelvin**
which has in ***NO WAY*** been measured ***ANYHOW***.

Appendix I — $\psi_{\text{approx}}(q)$ coefficients of ring

$c_k =$
```
[0
,2
,3/2
,45/32
,175/128
,11025/8192
,43659/32768
,693693/524288
,2760615/2097152
,703956825/536870912
,2807136475/2147483648
,44801898141/34359738368
,178837328943/137438953472
,11425718238025/8796093022208
,45635265151875/35184372088832
,729232910488125/562949953421312
,2913690606794775/2251799813685248
,2980705490751054825/2305843009213693952
,11912508103174630875/9223372036854775808
,190453061649520333125/147573952589676412928
,761284675790187924375/590295810358705651712
,48691767863540419643025/37778931862957161709568
,194656659282135509820075/151115727451828646838272
,3112897815792828194230125/2417851639229258349412352
,12445706768245428413604375/9671406556917033397649408
,3184718076363246848503430625/2475880078570760549798248448
,12733776756530806199056117011/9903520314283042199192993792
,203665080313034137018039551957/158456325028528675187087900672
,814380945284628956663354312695/633825300114114700748351602688
,52103760478924730186522770822425/40564819207303340847894502572032
,208353087384808403825536359424275/162259276829213363391578010288128
,3332723384435224201636023811413181/2596148429267413814265248164610048
,13327425563355428311016898550739703/10384593717069655257060992658440192
,54575807681940478933614199565279083785/42535295865117307932921825928971026432
,218253115201883182512295536369871818075/170141183460469231731687303715884105728
,3491294642139466964547897109473762681525/2722258935367507707706996859454145691648
,13962328532115305028016447297397521123911/10889035741470030830827987437816582766592
,893416651629057110619867238795201876360873/696898287454081973172991196020261297061888
,3573014001219202104195597613151008234533075/2787593149816327892691964784081045188247552
,57158326473797485184846471512318760538583125/44601490397061246283071436545296723011960832
];
```
Don't blame me...

The last entry equals: $\sim 1.281\,534\,001\,777\,7432$

& it ***seems*** they are approaching: $\frac{4}{\pi} \approx 1.273\,239\,544\,735\,1627$

which we could use to add more terms if desired.

Especially very close to the ring, this might improve the result.

Appendix II — $g^{*}_{\mathrm{approx}}(q_{\mathrm{disk}})$ coefficients of disk

$C_k =$
```
[0
,3.141592653589793
,1.1780972450961724
,0.7363107781851078
,0.5368932757599744
,0.42280345466097985
,0.3488128500953083
,0.2969061759739827
,0.258467429977351
,0.22885137029244623
,0.20533053501238924
,0.18619746243168933
,0.17032836051989764
,0.1569532168252262
,0.14552667768822483
,0.13565165313080957
,0.1270321210047894
,0.1194428858344665
,0.1127095858977196
,0.10669511238124772
,0.10129016260930294
,0.09640652976921153
,0.09197224674411254
,0.0879280125740404
,0.08422452291399247
,0.08082044844621863
,0.07768088487196167
,0.07477615092909985
,0.0720808465470655
,0.06957310773554506
,0.06723401187202242
,0.06504709912027115
,0.06299798435463762
,0.061074040230170705
,0.059264136631549336
,0.05755842513606147
,0.05594815967094546
,0.05442554646668736
,0.05298361787075273
,0.05161612570910881
];
```

A reasonably accurate extension would be achieved by: $C_k = \frac{4}{2k}$.
Especially very near the edge of the disk, this might improve the result.

