In https://www.lehman.edu/faculty/anchordoqui/chapter25.pdf we find:

reduced mass: $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \therefore \mu = \frac{m_1 m_2}{m_1+m_2}$ $\quad \therefore m_1 m_2 = \mu(m_1 + m_2) =: M\mu$

semilatus rectum: $r_0 = \frac{L^2}{\mu G m_1 m_2}$ $\quad = \frac{L^2}{GM\mu^2}$

energy: $E = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}$ $\quad = \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r}$

note: $L = \mu v r$ $\quad \therefore \frac{L^2}{2\mu r^2} = \frac{1}{2}\mu v^2,\ E = E_{\text{kin}} + E_{\text{pot}}$

$E < 0 \rightarrow$ elliptical orbit, $E = 0 \rightarrow$ parabolic orbit, $E > 0 \rightarrow$ hyperbolic orbit

(see also: http://henk-reints.nl/astro/HR-ISCO-photon-sphere-by-SR-only.pdf
under "derivation of the ISCO from")

excentricity: $e = \sqrt{1 + \frac{2EL^2}{\mu(G m_1 m_2)^2}}$ $\quad = \sqrt{1 + \frac{2EL^2}{\mu(GM)^2\mu^2}}$

orbit equation: $r = \frac{r_0}{1 - e\cos\theta}$

Sem. rect. of ellipse: $r_0 = a(1 - e^2)$ $\quad \therefore a = \frac{L^2}{GM\mu^2(1-e^2)}$

With: $\mathcal{L} := \frac{L}{\mu}$ (dimension: surface area per time)

and: $\mathcal{E} := \frac{E}{\mu}$ (dimension: squared velocity)

as well as: $\mathcal{M} := GM$ (dimension: volume per squared time)

we obtain: $e = \sqrt{1 + \frac{2\mathcal{E}\mathcal{L}^2}{\mathcal{M}^2}}$ $\quad \therefore 1 - e^2 = \frac{-2\mathcal{E}\mathcal{L}^2}{\mathcal{M}^2}$

and: $r = \frac{\mathcal{L}^2}{\mathcal{M}(1 - e\cos\theta)}$

as well as: $a = \frac{\mathcal{L}^2}{\mathcal{M}(1-e^2)} = \frac{\mathcal{L}^2}{\mathcal{M}} \cdot \frac{\mathcal{M}^2}{-2\mathcal{E}\mathcal{L}^2} \therefore$ $\quad \boldsymbol{a = \frac{-\mathcal{M}}{2\mathcal{E}}}$

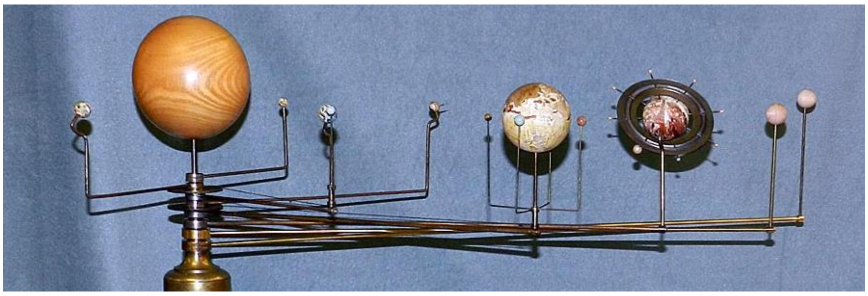
We also have: $b = a\sqrt{(1 - e^2)} = \left(\frac{-\mathcal{M}}{2\mathcal{E}}\sqrt{\frac{-2\mathcal{E}\mathcal{L}^2}{\mathcal{M}^2}}\right) = \frac{-\mathcal{L}}{2\mathcal{E}}\sqrt{-2\mathcal{E}}$

We define: $\mathscr{v} := \sqrt{-2\mathcal{E}}$ $\quad \therefore 2\mathcal{E} = -\mathscr{v}^2$ $\quad \boldsymbol{a = \frac{\mathcal{M}}{\mathscr{v}^2},\ b = \frac{\mathcal{L}}{\mathscr{v}}}$

(not *the* orbital velocity, but sort of; $\mathscr{v}$ decreases as r increases)

yielding "roundness": $\boldsymbol{\frac{b}{a}} = \sqrt{(1 - e^2)} = \sqrt{\frac{-2\mathcal{E}\mathcal{L}^2}{\mathcal{M}^2}}$ $\quad \boldsymbol{= \frac{\mathcal{L}\mathscr{v}}{\mathcal{M}}}$

and surface area: $\boldsymbol{A} = \pi ab = \pi \cdot \frac{\mathcal{M}}{\mathscr{v}^2} \cdot \frac{\mathcal{L}}{\mathscr{v}}$ $\quad \boldsymbol{= \frac{\pi\mathcal{M}\mathcal{L}}{\mathscr{v}^3}}$



https://www2.humboldt.edu/scimus/MedInst_rap/Orrery/Orrerya_12.jpg

