https://en.wikipedia.org/wiki/Conic_section
& https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections
use next general form of a conic section:

$$\boldsymbol{A}x^2 + \boldsymbol{B}xy + \boldsymbol{C}y^2 + \boldsymbol{D}x + \boldsymbol{E}y + \boldsymbol{F} = 0 \quad [1]$$

but https://en.wikipedia.org/wiki/Degenerate_conic
uses:

$$\boldsymbol{A}x^2 + \boldsymbol{2B}xy + \boldsymbol{C}y^2 + \boldsymbol{2D}x + \boldsymbol{2E}y + \boldsymbol{F} = 0 \quad [2]$$

and https://en.wikipedia.org/wiki/Hyperbola#Quadratic_equation
uses:

$$\boldsymbol{A_{xx}}x^2 + \boldsymbol{2A_{xy}}xy + \boldsymbol{A_{yy}}y^2 + \boldsymbol{2B_x}x + \boldsymbol{2B_y}y + \boldsymbol{C} = 0 \quad [3]$$

**How nicely consistent, dear doltish & dull dunces daring *d*o devise *d*ese damn different descriptions!**

I prefer [1], which seems the most basic form with only single-letter coefficients and variables.

In [3], we'll substitute: $\boldsymbol{A_{xx}} \Rightarrow \boldsymbol{A}$, $\boldsymbol{A_{xy}} \Rightarrow \boldsymbol{B/2}$, $\boldsymbol{A_{yy}} \Rightarrow \boldsymbol{C}$, $\boldsymbol{B_x} \Rightarrow \boldsymbol{D/2}$, $\boldsymbol{B_y} \Rightarrow \boldsymbol{E/2}$, $\boldsymbol{C} \Rightarrow \boldsymbol{F}$

For a **hyperbola**, the last aforementioned link gives next set of equations:

$$\begin{aligned}
\boldsymbol{A_{xx}} &= -a^2 \sin^2\theta + b^2\cos^2\theta \\
\boldsymbol{A_{yy}} &= -a^2 \cos^2\theta + b^2\sin^2\theta \\
\boldsymbol{A_{xy}} &= (a^2 + b^2)\sin\theta\cos\theta \\
\boldsymbol{B_x} &= -A_{xx}x_c - A_{xy}y_c \\
\boldsymbol{B_y} &= -A_{xy}x_c - A_{yy}y_c \\
\boldsymbol{C} &= A_{xx}x_c^2 + 2A_{xy}x_cy_c + A_{yy}y_c^2 - a^2b^2
\end{aligned} \quad [4]$$

which we'll convert from [3] to [1].

We obtain (in alphabetic order):

$$\begin{aligned}
\boldsymbol{A} &= -a^2\sin^2\theta + b^2\cos^2\theta \\
\boldsymbol{B} &= 2(a^2 + b^2)\sin\theta\cos\theta \\
\boldsymbol{C} &= -a^2\cos^2\theta + b^2\sin^2\theta \\
\boldsymbol{D} &= -2Ax_c - By_c \\
\boldsymbol{E} &= -Bx_c - 2Cy_c \\
\boldsymbol{F} &= Ax_c^2 + Bx_cy_c + Cy_c^2 - a^2b^2
\end{aligned} \quad [5]$$

For an **ellipse**, https://en.wikipedia.org/wiki/Ellipse#General_ellipse

gives:

$$\begin{aligned}
\boldsymbol{A} &= a^2\sin^2\theta + b^2\cos^2\theta \\
\boldsymbol{B} &= 2(b^2 - a^2)\sin\theta\cos\theta \\
\boldsymbol{C} &= a^2\cos^2\theta + b^2\sin^2\theta \\
\boldsymbol{D} &= -2Ax_c - By_c \\
\boldsymbol{E} &= -Bx_c - 2Cy_c \\
\boldsymbol{F} &= Ax_c^2 + Bx_cy_c + Cy_c^2 - a^2b^2
\end{aligned} \quad [6]$$

as well as:

$$\begin{aligned}
a, b &= \frac{-\sqrt{2(AE^2 + CD^2 - BDE + F[B^2 - 4AC])\cdot\left(A + C \pm \sqrt{(A-C)^2 - B^2}\right)}}{B^2 - 4AC} \\
x_c &= \frac{2CD - BE}{B^2 - 4AC} \\
y_c &= \frac{2AE - BD}{B^2 - 4AC} \\
\theta &= \tfrac{1}{2}\operatorname{atan2}(-B, C - A)
\end{aligned} \quad [7]$$

We need something similar for a hyperbola, which cannot easily be found on said hyperbola page.
**Nice, such dissimilar pages about essentially the same, dear authors!** ***Experts? Teachers? Professors?***
*The universe mainly consists of electrons, protons, neutrons & morons.*

  

But, the page *does* appear to give next.

It defines: $DD := \begin{vmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{vmatrix} = \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix} = AC - B^2/4$ [8]

(it is actually defined as $D$, but that conflicts with the targetted coefficient names).

as well as: $\Delta := \begin{vmatrix} A_{xx} & A_{xy} & B_x \\ A_{xy} & A_{yy} & B_y \\ B_x & B_y & C \end{vmatrix} = \begin{vmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{vmatrix}$

$= ACF + (BDE - AE^2 - B^2F - CD^2)/4$ [9]

and: $x_c = \frac{-1}{DD}\begin{vmatrix} B_x & A_{xy} \\ B_y & A_{yy} \end{vmatrix} = \begin{vmatrix} D/2 & B/2 \\ E/2 & C \end{vmatrix} = \frac{CD/2 - BE/4}{AC - B^2/4} = \frac{2CD - BE}{4AC - B^2}$

$y_c = \frac{-1}{DD}\begin{vmatrix} A_{xx} & B_x \\ A_{xy} & B_y \end{vmatrix} = \begin{vmatrix} A & D/2 \\ B/2 & E/2 \end{vmatrix} = \frac{AE/2 - BD/4}{AC - B^2/4} = \frac{2AE - BD}{4AC - B^2}$ [10]

plus: $\tan(2\varphi) = \tan(2\theta)? = \frac{2A_{xy}}{A_{xx} - A_{yy}} = \frac{B}{A - C}$

hence: $\theta = \frac{1}{2}\operatorname{atan2}(B, A - C)$ [11]

Then it says: $a^2 = \frac{-\Delta}{\lambda_1^2 \lambda_2}$

as well as: $b^2 = \frac{-\Delta}{\lambda_1 \lambda_2^2}$ [12]

where: $\lambda_{1,2} =$ roots of $\left[\lambda - \left(A_{xx} + A_{yy}\right)\lambda + DD = 0\right] = [\lambda - (A + C)\lambda + AC - B^2/4 = 0]$

hence: $\lambda_{1,2} = \frac{A + C \pm \sqrt{(A+C)^2 - 4 \cdot 1 \cdot (AC - B^2/4)}}{2 \cdot 1} = \frac{A + C \pm \sqrt{(A+C)^2 + B^2 - 4AC}}{2}$ [13]

For a **parabola**, https://en.wikipedia.org/wiki/Parabola#General_position
says:

if *directrix* given by: $ax + by + c = 0$

then: $\frac{(ax+by+c)^2}{a^2+b^2} = (x - x_f)^2 + (y - y_f)^2$ (sq.dist. to *directrix* = sq.dist. to *focus*).

hence: $a^2x^2 + b^2y^2 + c^2 + 2abxy + 2acx + 2bcy$

$= (a^2 + b^2)(x^2 + y^2 - 2xx_f - 2yy_f + x_f^2 + y_f^2)$

or: $\boldsymbol{a^2x^2} + \boldsymbol{b^2y^2} + c^2 + 2abxy + 2acx + 2bcy$

$= \boldsymbol{a^2x^2} + a^2y^2 - 2a^2xx_f - 2a^2yy_f + a^2x_f^2 + a^2y_f^2$

$+ b^2x^2 + \boldsymbol{b^2y^2} - 2b^2xx_f - 2b^2yy_f + b^2x_f^2 + b^2y_f^2$

yielding: $c^2 + 2abxy + 2acx + 2bcy$

$= a^2y^2 - 2a^2x_fx - 2a^2y_fy + a^2x_f^2 + a^2y_f^2$

$+ b^2x^2 - 2b^2x_fx - 2b^2y_fy + b^2x_f^2 + b^2y_f^2$

i.e.:

$$
\begin{aligned}
& b^2x^2 \\
- \; & 2abxy \\
+ \; & a^2y^2 \\
- \; & 2(a^2x_f + b^2x_f + ac)x \\
- \; & 2(a^2y_f + b^2y_f + bc)y \\
+ \; & (a^2 + b^2)x_f^2 + (a^2 + b^2)y_f^2 - c^2 \\
= \; & 0
\end{aligned}
$$ [14]

  

which has the form of [1], from which we conclude:

$$\begin{aligned} A &= b^2 \\ B &= -2ab \\ C &= a^2 \\ D &= -2(Cx_\mathrm{f} + Ax_\mathrm{f} + ac) = -2\big((C+A)x_\mathrm{f} + ac\big) \\ E &= -2(Cy_\mathrm{f} + Ay_\mathrm{f} + bc) = -2\big((C+A)y_\mathrm{f} + bc\big) \\ F &= (C+A)x_\mathrm{f}^2 + (C+A)y_\mathrm{f}^2 - c^2 \end{aligned} \qquad [15]$$

where: $\{a, b, c\}$ specifies the *directrix*: $ax + by + c = 0$

and: $(x_\mathrm{f}, y_\mathrm{f})$ is the *focus*.

We have: $ax + by + c = 0 \therefore y = -\frac{a}{b}x - \frac{c}{b}$

slope & orientation: $\tan\varphi = -\frac{a}{b} \therefore \varphi = \operatorname{atan2}(-a, b)$

then the orientation of the parabola's axis of symmetry

equals: $\theta = \varphi + \frac{\pi}{2}$

and, using: $\tan\theta = \tan\left(\varphi + \frac{\pi}{2}\right) = \frac{-1}{\tan\varphi} = \frac{b}{a}$

we obtain: $\theta = \operatorname{atan2}(b, a)$ which is (or should be) normalised to: $\theta \in (-\pi, \pi]$ .

The focal length $f$ (distance from *focus* to *vertex*) equals half the *semi-latus rectum* $p$ (distance from *focus* to parabola, parallel to *directrix*), which in turn equals the distance from *focus* to *directrix*:

$$p = 2f = \frac{ax_\mathrm{f} + by_\mathrm{f} + c}{\sqrt{a^2 + b^2}}$$

For *distances*, we should actually use *absolute values*, but instead, we'll use $p > 0$ as a *requirement*. A negative $p$ would refer to the parabola's mirror image on the other side of the directrix (*when looking into a mirror in order to comb your hair or so, you are definitely not watching yourself or whatever living person, but merely a mirrored (i.e. malformed!) image thereof, ceçi nest pas une pipe*). When deriving $\{a, b, c\}$ from $\{A, B, C, D, E, F\}$, we are solving a set of quadratic equations and we should then discard any solution that renders $p \le 0$ .

All to get her[1], we can fully characterise any parabola by: $\boldsymbol{\{x_\mathrm{f}, y_\mathrm{f}, p, \theta\}}$ (*position*, *size* & *orientation*).

We derive the *directrix* as: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_\mathrm{f} \\ y_\mathrm{f} \end{bmatrix} - p\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \lambda\begin{bmatrix} \sin(\theta - \pi/2) \\ \cos(\theta - \pi/2) \end{bmatrix} = \begin{bmatrix} x_\mathrm{f} - p\cos\theta \\ y_\mathrm{f} - p\sin\theta \end{bmatrix} + \lambda\begin{bmatrix} -\cos\theta \\ +\sin\theta \end{bmatrix}$

Its direction vector: $\begin{bmatrix} -\cos\theta \\ +\sin\theta \end{bmatrix}$

yields the normal vector: $\begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$

$\therefore$ *directrix*: $x\sin\theta + y\cos\theta - c = 0 \therefore \boldsymbol{a = \sin\theta}, \ \boldsymbol{b = \cos\theta}$ [a]

Substituting the position vector: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_\mathrm{f} - p\cos\theta \\ y_\mathrm{f} - p\sin\theta \end{bmatrix}$

then renders: $(x_\mathrm{f} - p\cos\theta)\sin\theta + (y_\mathrm{f} - p\sin\theta)\cos\theta - c = 0$

yielding: $c = x_\mathrm{f}\sin\theta - p\cos\theta\sin\theta + y_\mathrm{f}\cos\theta - p\sin\theta\cos\theta$

or: $c = x_\mathrm{f}\sin\theta + y_\mathrm{f}\cos\theta - 2p\sin\theta\cos\theta$

so, using [a]: $c = x_\mathrm{f}a + y_\mathrm{f}b - 2pab$

**so, overall:** $\boldsymbol{a = \sin\theta}, \ \boldsymbol{b = \cos\theta}, \ \boldsymbol{c = x_\mathrm{f}a + y_\mathrm{f}b - 2pab}$ [16]

By substituting [16] into [15]

we obtain:

$$\begin{aligned} A &= b^2 && = \cos^2\theta \\ B &= -2ab && = -2\sin\theta\cos\theta = -\sin(2\theta) \\ C &= a^2 && = \sin^2\theta \end{aligned}$$

**N.B.:** $\boldsymbol{C + A = a^2 + b^2 = \sin^2\theta + \cos^2\theta = 1}$

[1] Oops, I maid a miss stake...

  

$$
\begin{aligned}
D &= -2(Cx_\mathrm{f} + Ax_\mathrm{f} + ac) &&= -2(x_\mathrm{f} + ac)\\
E &= -2(Cy_\mathrm{f} + Ay_\mathrm{f} + bc) &&= -2(y_\mathrm{f} + bc)\\
F &= (C + A)x_\mathrm{f}^2 + (C + A)y_\mathrm{f}^2 - c^2 &&= x_\mathrm{f}^2 + y_\mathrm{f}^2 - c^2
\end{aligned} \qquad [17]
$$

Note: we obviously have:

vertical parabola: $\theta = \pm\pi/2 \quad \therefore \cos\theta = b = 0 \quad \therefore A = 0,\ B = 0$

horizontal parabola: $\theta \in \{0, \pi\} \quad \therefore \sin\theta = a = 0 \quad \therefore C = 0,\ B = 0$

hence:

$$
\begin{aligned}
A &= b^2\\
B &= -2ab\\
C &= a^2\\
D &= -2x_\mathrm{f} - 2ac\\
E &= -2y_\mathrm{f} - 2bc\\
F &= x_\mathrm{f}^2 + y_\mathrm{f}^2 - c^2
\end{aligned} \qquad [18]
$$

This nicely fulfills: $B^2 - 4AC = 0$ (see https://en.wikipedia.org/wiki/Parabola#General_position)

Now we want to reverse [16] and derive $\{a, b, c, x_\mathrm{f}, y_\mathrm{f}\}$ .

As said, we require: $p = \frac{ax_\mathrm{f}+by_\mathrm{f}+c}{\sqrt{a^2+b^2}} = \boldsymbol{ax_\mathrm{f} + by_\mathrm{f} + c > 0}$ (found above: $a^2 + b^2 = 1$)

& we easily find: $\tan^2\theta = C/A \therefore \theta = \mathrm{atan2}(\pm\sqrt{C}, \pm\sqrt{A})$ (still have to find correct quadrant)

as well as: $\sin(2\theta) = -B$ , but weren't we looking for $\{a, b, c, x_\mathrm{f}, y_\mathrm{f}\}$ and not $\theta$ ?

Of course the quadratic behaviour is a kind of bottleneck, but we'll give it a try:

$$
\begin{aligned}
&C \ \therefore\ a = \pm\sqrt{C}\\
&A \ \therefore\ b = \pm\sqrt{A}\\
&F \ \therefore\ c^2 = F - x_\mathrm{f}^2 - y_\mathrm{f}^2\\
&D \ \therefore\ x_\mathrm{f} = -D/2 - (\pm\sqrt{C})\left(\pm\sqrt{F - x_\mathrm{f}^2 - y_\mathrm{f}^2}\right) = -D/2 \pm \sqrt{C}\sqrt{F - x_\mathrm{f}^2 - y_\mathrm{f}^2} = -D/2 \pm c\sqrt{C}\\
&E \ \therefore\ y_\mathrm{f} = -E/2 - (\pm\sqrt{A})\left(\pm\sqrt{F - x_\mathrm{f}^2 - y_\mathrm{f}^2}\right) = -E/2 \pm \sqrt{A}\sqrt{F - x_\mathrm{f}^2 - y_\mathrm{f}^2} = -E/2 \pm c\sqrt{A}
\end{aligned} \qquad [b]
$$

Required: $\boldsymbol{ax_\mathrm{f} + by_\mathrm{f} + c > 0} \therefore \pm\sqrt{C}(-D/2 \pm c\sqrt{C}) \pm \sqrt{A}(-E/2 \pm c\sqrt{A}) + c > 0$

i.e.: $\pm(-D/2)\sqrt{C} \pm c\sqrt{C}\sqrt{C} \pm (-E/2)\sqrt{A} \pm c\sqrt{A}\sqrt{A} + c > 0$

so: $\pm D\sqrt{C}/2 \pm E\sqrt{A}/2 + c \pm cC \pm cA > 0$

or: $\boldsymbol{\pm D\sqrt{C}/2 \pm E\sqrt{A}/2 + c(1 \pm C \pm A) > 0}$ (cannot use $C + A = 1$ )

[b] says: $c^2 = F - \left(-D/2 \pm c\sqrt{C}\right)^2 - \left(-E/2 \pm c\sqrt{A}\right)^2$

$= F - \left(D^2/4 \pm 2(D/2)c\sqrt{C} + c^2C\right) - \left(E^2/4 \pm 2(E/2)c\sqrt{A} + c^2A\right)$

$= F - D^2/4 \pm cD\sqrt{C} - c^2C - E^2/4 \pm cE\sqrt{A} - c^2A$

so: $c^2 + c^2C + c^2A \pm cD\sqrt{C} \pm cE\sqrt{A} + D^2/4 + E^2/4 = 0$

i.e.: $(1 + C + A)c^2 + (\pm D\sqrt{C} \pm E\sqrt{A})c + (D^2 + E^2)/4 = 0$

yielding: $2c^2 + (\pm D\sqrt{C} \pm E\sqrt{A})c + (D^2 + E^2)/4 = 0$ (used $C + A = 1$ )

From this, we obtain:

$$
\begin{aligned}
c &= \frac{-(\pm D\sqrt{C} \pm E\sqrt{A}) \pm \sqrt{\left(\pm D\sqrt{C} \pm E\sqrt{A}\right)^2 - 4\cdot 2\cdot(D^2+E^2)/4}}{2\cdot 2} \qquad \text{(lower case "}c\text{" left of "=")}\\
&= \frac{\pm D\sqrt{C} \pm E\sqrt{A} \pm \sqrt{\left(\pm D\sqrt{C}\right)^2 + 2\left(\pm D\sqrt{C}\right)\left(\pm E\sqrt{A}\right) + \left(\pm E\sqrt{A}\right)^2 - 2(D^2+E^2)}}{4}\\
&= \frac{\pm D\sqrt{C} \pm E\sqrt{A} \pm \sqrt{D^2C \pm 2DE\sqrt{AC} + E^2A - 2D^2 - 2E^2}}{4}
\end{aligned}
$$

We already found: $a = \pm\sqrt{C}$ & $b = \pm\sqrt{A}$ which we substitute into this equation, thus eliminating some "$\pm$" options, which leaves us

with: $\boldsymbol{a = \pm\sqrt{C}}$

$\boldsymbol{b = \pm\sqrt{A}}$

and: $\boldsymbol{c=\frac{aD+bE\pm\sqrt{D^2(C-2)+E^2(A-2)\pm 2abDE}}{4}}$

& once we know the correct value of $c$, we can easily find:

$$x_f = -D/2 \pm c\sqrt{C} = -D/2 \pm ac$$

and: $$y_f = -E/2 \pm c\sqrt{A} = -E/2 \pm bc$$

We now have a total of six "$\pm$" choices, rendering $2^6 = 64$ possible *directrix* & *focus* combinations, which should all be checked for fulfilling said requirement of $ax_f + by_f + c > 0$. In a computer program this should not be a big problem, but analytically it seems a hell of a job. Hopefully, only one true solution will emerge.

Instead of checking this requirement, said computer program can also use [18] to rederive $\{A, B, C, D, E, F\}$ from each possible $\{a, b, c, x_f, y_f\}$ and then it SHOULD be that only one of the latter will reproduce the original $\{A, B, C, D, E, F\}$, given some tolerance that is inherent to floating point arithmetic. It could simply use the one that is closest (using 6D Pythagoras!) to the original and, once again, there will hopefully be just one such solution, which should have a really small 6D distance compared to all others.

(Sigh).

Some mathematicians claim there is only one parabola,
but isn't a proposition immediately falsified
by merely one single counterexample?
Well, here it is:

*Two parabolas (and their focuses).*