How much is i^i ?

See also: https://henk-reints.nl/HR-introductie-complexe-getallen.pdf (which is in Dutch).

Since: $\forall z \neq 0$: $z = e^{\ln z}$, we can say: $z^w = \left(e^{\ln z}\right)^w = e^{w \ln z} = e^{\ln z^w}$,
hence safely change from any non-zero base to base e & therefore:

$$i^i = e^{\ln i^i} = e^{i \ln i} = \cos \ln i + i \sin \ln i$$

We will not directly resolve this $\ln i$ (see p.2 for the formula),
but try to find φ such that: $e^{i\varphi} = i \therefore \ln i = i\varphi$

$$\therefore e^{i\varphi} = \cos \varphi + i \sin \varphi = i$$

Easy peasy, lemon squeezy:

$$\cos \varphi = 0 \,, \ \sin \varphi = 1$$

$$\therefore \varphi = \frac{\pi}{2} + k \cdot 2\pi$$ *BINGO!* 👍

$$\therefore \ln i = i\varphi = \frac{\pi i}{2} + 2k\pi i$$

With this result, we obtain: $i^i = \cos i\varphi + i \sin i\varphi$

$$\therefore i^i = \cosh \varphi + i^2 \sinh \varphi = \cosh \varphi - \sinh \varphi$$

$$\therefore i^i = \frac{e^{\varphi} + e^{-\varphi}}{2} - \frac{e^{\varphi} - e^{-\varphi}}{2} = \frac{2e^{-\varphi}}{2} = e^{-\varphi}$$

$$\therefore i^i = e^{-\varphi} = e^{-\left(\frac{\pi}{2} + 2k\pi\right)}$$

Since $k \in \mathbb{Z}$, this is an infinite list of values.
We'll concentrate on the *principal value*:

$$i^i = e^{-\pi/2} \approx 0.207\,880$$

Q.E.D.

Did you imagine this is a **real** value?

i, the unimaginable imaginary,
i imagine i can unimaginably imaginarily rise to
the unimaginable reality behind imaginary numbers.

How much is $\ln z$?

We'll start with: $z = re^{i\varphi}$

where of course: $r = |z|$ & $\varphi = \arg z$

But can't we circle around O and obtain the same?

Hence: $z = re^{i(\varphi+2k\pi)}$

MEMORISE:

***ALWAYS** circle around with complex numbers!*

We obtain: $\ln z = \ln re^{i(\varphi+2k\pi)}$

All rules of logarithms also apply to complex logarithms, such as *log of product equals sum of logs of factors*, so:

$$\ln re^{i(\varphi+2k\pi)} = \ln r + \ln e^{i(\varphi+2k\pi)}$$

which yields: $\ln z = \ln|z| + i(\varphi + 2k\pi)$

Usually/frequently, $+2k\pi$ in the final result does not render any new values, but this time, it ***DOES***, *do you see that?*

Therefore:

$$\ln z = \ln|z| + i(\arg z + 2k\pi)$$

Q.E.D.

Since this renders multiple results for a single input value, **the *complex logarithm* is *not a function*!**

In practice however, we can sometimes/often find a good reason to discard this $2k\pi$.

The ***principle value*** $\mathrm{Log}\, z$ is defined as that part of the complex logarithm that has:

$$-\pi < \mathrm{Im}(\ln z) \leq \pi$$

i.e.: $\mathrm{Log}\, z = \ln|z| + i \arg z$ with $\arg z \in (-\pi, \pi]$

$\mathrm{Log}\, z$ ***IS*** a function. *One input yields one result.*