Suppose a sequence defined by:	$\boldsymbol{F_1, F_2 \in \mathbb{N}}$ $\boldsymbol{\forall\{(n \in \mathbb{N}) > 2\}: F_{n+1} = F_n + F_{n-1}}$
which implies:	$\forall\{n \in \mathbb{N}\}: F_n \in \mathbb{N}$
Ratio of consecutive values:	$r_{n+1} = \frac{F_{n+1}}{F_n}$
Please note: $\boldsymbol{r_1}$ is ***undefined***, since there exists no $\boldsymbol{F_0}$ **!**	
obviously, since $F_n, F_{n+1} \in \mathbb{N}$:	$r_{n+1} > 0$
We have:	$r_{n+1} = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$
therefore:	$\boldsymbol{r_{n+1} = 1 + \frac{1}{r_n}}$
IFF this ratio sequence has:	$\lim_{n \to \infty} r_n = L$
which of course applies to both:	$\boldsymbol{r_n}$ and $\boldsymbol{r_{n+1}}$
then it must be that:	$L = 1 + \frac{1}{L}$
i.e.:	$L^2 = L + 1$
which is:	$L^2 - L - 1 = 0$
yielding:	$L = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$

which of course is:	$L = \frac{1 \pm \sqrt{5}}{2}$
because:	$F_n \in \mathbb{N} \therefore r_n \in \mathbb{N} \therefore L \in \mathbb{N}$
this leaves:	$\boldsymbol{L = \frac{1+\sqrt{5}}{2} = \varphi}$

IFF this *ratio sequence* has a limit, it ***IS*** the **Golden Ratio**!

Had we ***chosen***: $F_2 = F_1 = 1$, then the main sequence would have been the **Fibonacci sequence**:

$\{\mathbf{1}, \mathbf{1}, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots\}$

This **Golden Ratio** as a *sequence ratio limit* is, as just shown, fully ***independent*** of the two ***start values*** (which were nowhere used explicitly), so in this regard, the **Fibonacci sequence** is **nothing special**!

By the way, Leonardo Pisano (Leonard from Pisa), whose father was named Bonacci, turning him into figlio di Bonacci ("*feel-yo dee bo-naht-chee*" = son of B.), derived it for the growth of a rabbit population under certain conditions (see Appendix I), which evidently renders Sweet Fanny Adams if you start with zero rabbits, so the Fibonacci sequence **does definitely *not* start with** $\{\mathbf{0}, \mathbf{1}\}$, dunce!

Would one start with $\boldsymbol{F_1 = 2}$, $F_2 = 1$, the **Lucas Numbers** would appear:

$\{\mathbf{2}, \mathbf{1}, 3, 4, 7, 11, 18, 29, 47, 76, 123, \cdots\}$

BUT... we have *not* yet *proven* it *does* have a *limit* at all!

We had:	$r_{n+1} = 1 + \frac{1}{r_n}$

It is well-known that:	$\frac{1}{\varphi} = \varphi - 1$
hence:	$1 + \underset{>\varphi}{\frac{1}{\quad}} < 1 + \varphi - 1 < \varphi$
as well as:	$1 + \underset{<\varphi}{\frac{1}{\quad}} > 1 + \varphi - 1 > \varphi$
and of course:	$1 + \frac{1}{\varphi} = \varphi$
therefore:	$r_n < \varphi \Rightarrow r_{n+1} > \varphi$
and:	$r_n > \varphi \Rightarrow r_{n+1} < \varphi$
or maybe: which would mean:	$r_n = \varphi \Rightarrow r_{n+1} = \varphi$ r_{n++} always remains: φ . Q.E.D.
Ignoring this last option, we now know r is alternatingly ***greater than*** and ***less than*** the **Golden Ratio**!	
Now we only have to prove: $\lvert r - \varphi \rvert$ is monotonically decreasing. It would also render/confirm the already proven value of: $L = \varphi$.	
We equate/define:	$r_k =: \varphi + \varepsilon_k$
then:	$\varphi + \varepsilon_{n+1} = 1 + \frac{1}{\varphi + \varepsilon_n} = \frac{\varphi + \varepsilon_n + 1}{\varphi + \varepsilon_n}$

hence:	$\varepsilon_{n+1} = \frac{\varphi+\varepsilon_n+1}{\varphi+\varepsilon_n} - \varphi = \frac{\varphi+\varepsilon_n+1-\varphi(\varphi+\varepsilon_n)}{\varphi+\varepsilon_n}$ $= \frac{\varphi-\varphi^2+\varepsilon_n+1-\varphi\varepsilon_n}{\varphi+\varepsilon_n} = \frac{\varphi-(\varphi+1)+\varepsilon_n+1-\varphi\varepsilon_n}{\varphi+\varepsilon_n}$ $= \frac{-1+\varepsilon_n+1-\varphi\varepsilon_n}{\varphi+\varepsilon_n} = \frac{\varepsilon_n-\varphi\varepsilon_n}{\varphi+\varepsilon_n} = \frac{\varepsilon_n(1-\varphi)}{\varphi+\varepsilon_n}$
This renders:	$\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{1-\varphi}{\varphi+\varepsilon_n} = \frac{-1/\varphi}{\varphi+\varepsilon_n} = \frac{-1}{\varphi^2+\varphi\varepsilon_n} = \frac{-1}{1+\varphi+\varphi\varepsilon_n}$ $\therefore \frac{-\varepsilon_{n+1}}{\varepsilon_n} = \frac{1}{\mathbf{1+\varphi+\varphi\varepsilon_n}}$
Due to the found alternation, we have:	$\frac{\varepsilon_{n+1}}{\varepsilon_n} < 0 \therefore \frac{-\varepsilon_{n+1}}{\varepsilon_n} = \left\lvert\frac{\varepsilon_{n+1}}{\varepsilon_n}\right\rvert$
For monotonic decrease by absolute value, we ***NEED***:	$0 < \left\lvert\frac{\varepsilon_{n+1}}{\varepsilon_n}\right\rvert = \frac{1}{1+\varphi+\varphi\varepsilon_n} < 1$
which demands:	$1 + \varphi + \varphi\varepsilon_n > 1$
yielding next ***requirement***, to be met for ***some*** n:	$\varphi\varepsilon_n > 1 - (1+\varphi) = -\varphi$ $\therefore \varepsilon_n > -1$
We have:	$r_2 = \frac{F_2}{F_1} > 0$ ***can*** *be* < 1 .
But for $n > 2$:	$r_{(n>2)+1} = 1 + \frac{1}{r_n} > 1$

so r has a ***lower bound***:	$B_{\text{low}} = 1$
rendering:	$\varepsilon_{\text{low}} = B_{\text{low}} - \varphi = 1 - \varphi > -1$
Also:	$F_{(n>2)+1} = F_n + F_{n-1}$
hence:	$F_{(n>2)} < F_{n+1}$ & $F_{n-1} < F_n$
& therefore:	$F_n + F_{n-1} < 2F_n$
so we obtain:	$r_{(n>2)+1} = \frac{F_{(n>2)+1}}{F_n} = \frac{F_n + F_{n-1} < 2F_n}{F_n} < 2$
yielding an ***upper bound***:	$B_{\text{upp}} = 2$
which implies:	$\varepsilon_{\text{upp}} = B_{\text{upp}} - \varphi = 2 - \varphi > -1$
Now both bounds meet the said ***requirement***,	
so we now know for sure:	$\varepsilon_{(n>2)+1} > -1$
implying:	$0 < \left\lvert \frac{\varepsilon_{(n>2)+1}}{\varepsilon_n} \right\rvert < 1$

This means the sequence *does* have a limit equal to φ !

Do you see ***no*** specific (*numerical*) start values were used for this proof?

Therefore, this limit applies to *ALL Fibonaccish* sequences

that follow the definition given at the very top of *this* very document.

& Clara est Cornelis.[1]

Quod Erat Demonstrandum!

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[1] Pseudo Latin version of Dutch expression: En klaar is Kees;
meaning: *we're all done*, *that's it*!

Appendix I

How on Earth did Fibonacci = figlio di Bonacci = Leonardo Pisano

(son of Bonacci, Leonard from Pisa, $(1170 \pm ?) - (1245 \pm 5?)$ CE) come to his sequence?

Well, he made next presumptions (never mind the biological complications…):

1. two months after its birth, a rabbit reaches adulthood;
2. from then on, it will once every month (until eternity!) get one baby;
3. the initial population is one single baby rabbit.

M.	**Rabbit population & events** (look at the colours)	**Remarks** (look at the colours)
0	$\{\boldsymbol{R_1}\}$	R_1 is a **baby**
1	$\{\boldsymbol{R_1}\}$	R_1 is an **adolescent**
2	$\{\boldsymbol{R_1}\} \rightarrow \{R_2\} \vdash \{\boldsymbol{R_1}, \boldsymbol{R_2}\}$	R_1 is an **adult** & gives birth to R_2
3	$\{\boldsymbol{R_1}\} \rightarrow \{R_3\} \vdash \{\boldsymbol{R_1}, \boldsymbol{R_2}, \boldsymbol{R_3}\}$	R_1 gets a baby, R_2 not yet
4	$\{R_1, \boldsymbol{R_2}\} \rightarrow \{R_4, R_5\} \vdash \{R_1, \boldsymbol{R_2}, \boldsymbol{R_3}, R_4, \boldsymbol{R_5}\}$	$\{R_1, R_2\}$ both get baby, R_3 not yet
5	$\{R_1 \cdots \boldsymbol{R_3}\} \rightarrow \{R_6, R_7, R_8\} \vdash \{R_1 \cdots \boldsymbol{R_3}, R_4, \boldsymbol{R_5}, R_6 \cdots \boldsymbol{R_8}\}$	$\{R_1 \cdots R_3\}$ get baby, $\{R_4, R_5\}$ not yet
6	$\{R_1 \cdots \boldsymbol{R_5}\} \rightarrow \{R_9 \cdots R_{13}\} \vdash \{R_1 \cdots \boldsymbol{R_5}, R_6 \cdots \boldsymbol{R_8}, R_9 \cdots \boldsymbol{R_{13}}\}$	*Do you grasp the systematics?*
7	$\{R_1 \cdots \boldsymbol{R_8}\} \rightarrow \{R_{14} \cdots R_{21}\} \vdash \{R_1 \cdots \boldsymbol{R_8}, R_9 \cdots \boldsymbol{R_{13}}, R_{14} \cdots \boldsymbol{R_{21}}\}$	*Watch greatest/rightmost index!*
8	$\{R_1 \cdots \boldsymbol{R_{13}}\} \rightarrow \{R_{22} \cdots R_{34}\} \vdash \{R_1 \cdots \boldsymbol{R_{13}}, R_{14} \cdots \boldsymbol{R_{21}}, R_{22} \cdots \boldsymbol{R_{34}}\}$	*See the sequence evolve?*
&cetera ad infinitum. **Baby ↗ adolescent ↗ adult** = monthly giving birth forever.		

Appendix II

Numerical evaluation of the Fibonnacci sequence; it approaches: $0.7236\varphi^{n-1}$

n	Fn	ratio	epsilon	PHI^(n-1)	Fn/PHI^(n-1)
1	1	NaN	NaN	1.0000000000000000	1.0000000000000000
2	1	1.0000000000000000	-0.618	1.6180339887498950	0.6180339887498948
3	2	2.0000000000000000	+0.382	2.6180339887498950	0.7639320225002103
4	3	1.5000000000000000	-0.118	4.2360679774997900	0.7082039324993691
5	5	1.6666666666666667	+4.86e-2	6.8541019662496850	0.7294901687515772
6	8	1.6000000000000000	-1.80e-2	1.1090169943749474e+1	0.7213595499957939
7	13	1.6250000000000000	+6.97e-3	1.7944271909999160e+1	0.7244651700109358
8	21	1.6153846153846154	-2.65e-3	2.9034441853748635e+1	0.7232789287212935
9	34	1.6190476190476190	+1.01e-3	4.6978713763747800e+1	0.7237320325750782
10	55	1.6176470588235294	-3.87e-4	7.6013155617496440e+1	0.7235589623033660
11	89	1.6181818181818181	+1.48e-4	1.2299186938124424e+2	0.7236250692647179
12	144	1.6179775280898876	-5.65e-5	1.9900502499874068e+2	0.7235998186523744
13	233	1.6180555555555555	+2.16e-5	3.2199689437998495e+2	0.7236094635280528
14	377	1.6180257510729614	-8.24e-6	5.2100191937872570e+2	0.7236057795133609
15	610	1.6180371352785145	+3.15e-6	8.4299881375871060e+2	0.7236071866817582
16	987	1.6180327868852460	-1.20e-6	1.3640007331374363e+3	0.7236066491912583
17	1597	1.6180344478216820	+4.59e-7	2.2069995468961470e+3	0.7236068544943606
18	2584	1.6180338134001253	-1.75e-7	3.5710002800335833e+3	0.7236067760755535
19	4181	1.6180340557275540	+6.70e-8	5.7779998269297300e+3	0.7236068060288725
20	6765	1.6180339631667064	-2.56e-8	9.3490001069633140e+3	0.7236067945877226
21	10946	1.6180339985218033	+9.77e-9	1.5126999933893046e+4	0.7236067989578530
22	17711	1.6180339850173580	-3.73e-9	2.4476000040856360e+4	0.7236067972886117
23	28657	1.6180339901755971	+1.43e-9	3.9602999974749410e+4	0.7236067979262051
24	46368	1.6180339882053250	-5.45e-10	6.4079000015605770e+4	0.7236067976826660
25	75025	1.6180339889579020	+2.08e-10	1.0368199999035518e+5	0.7236067977756897
26	121393	1.6180339886704431	-7.95e-11	1.6776100000596096e+5	0.7236067977401578
27	196418	1.6180339887802426	+3.03e-11	2.7144299999631615e+5	0.7236067977537297
28	317811	1.6180339887383030	-1.16e-11	4.3920400000227714e+5	0.7236067977485456
29	514229	1.6180339887543224	+4.43e-12	7.1064699999859330e+5	0.7236067977505257
30	832040	1.6180339887482035	-1.69e-12	1.1498510000008705e+6	0.7236067977497693
31	1346269	1.6180339887505408	+6.46e-13	1.8604979999994640e+6	0.7236067977500582
32	2178309	1.6180339887496482	-2.47e-13	3.0103490000003350e+6	0.7236067977499478
33	3524578	1.6180339887499890	+9.41e-14	4.8708469999997990e+6	0.7236067977499900
34	5702887	1.6180339887498590	-3.60e-14	7.8811960000001340e+6	0.7236067977499738
35	9227465	1.6180339887499086	+1.38e-14	1.2752042999999933e+7	0.7236067977499800
36	14930352	1.6180339887498895	-5.33e-15	2.0633239000000067e+7	0.7236067977499777
37	24157817	1.6180339887498970	+2.00e-15	3.3385282000000000e+7	0.7236067977499786
38	39088169	1.6180339887498940	-8.88e-16	5.4018521000000070e+7	0.7236067977499783
39	63245986	1.6180339887498951	+2.22e-16	8.7403803000000070e+7	0.7236067977499783
40	102334155	1.6180339887498946	-2.22e-16	1.4142232400000015e+8	0.7236067977499783
41	165580141	1.6180339887498950	+0.00	2.2882612700000024e+8	0.7236067977499782